Fluid Equations

Unknowns: $n_e, n_i, P_i, P_e, \mathbf{u_e}, \mathbf{u_i}, \mathbf{E}, \mathbf{B}$

first 4 are scalars the 4 following are vectors which makes 12 more for a total of:
16 unknowns

Equations:

a)Maxwell: $\cdot\ \mathbf{B} = 0$, $\times \mathbf{B} = \mu_0 \varepsilon_0 \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$, $\cdot\ \mathbf{D} = \rho$,

$\times\ \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ as pointed out these four vector equations are linearly independent so there is only two of them each. Thus for a) we have 2 vector equations X3 components = 6 equations

b) Continuity of mass $\frac{\partial p_m}{\partial t} + \cdot\ \left(\rho_m \mathbf{u}\right) = 0$ 2 species X 1 scalar equation = 2 equations

c) Force: $\rho_m \left(\left(\frac{\partial \mathbf{u}}{\partial t}\right) + (\mathbf{u} \cdot\ )\mathbf{u}\right) = -\ p + qn(\mathbf{E} + \mathbf{u} \times \mathbf{B})$ 3 equations/species X2 species = 6 equations

The Grand total is 14 equations

We need 2 more equations to close the fluid set. These must be the equations of state that relate the Thermodynamic variables. Which equation we choose is dictated by the problem and is an assumption. (The use of fluid theory is also an assumption)

Some equations of state are:

1) $P = C\rho_m^{\gamma}$ adiabatic, $\gamma = c_p/c_v$ ratio of specific heats at constant volume and at constant pressure. (May be written $\frac{\partial}{\partial t}\left(P\rho_m^{-\gamma}\right) = 0$)

2) $\cdot\ \mathbf{v} = 0$ incompressible fluid

3) Isothermal plasma $\frac{\partial}{\partial t}\left(\frac{P}{\rho_m}\right) = 0$

Each choice adds two more equations which closes the loop

It is sometimes useful to sum the equations over the two fluids and get an equation for the "total plasma fluid". Note here the mass density is :

$\rho_m = n_{ion} M_{ion} + n_e m_e = n(M_{ion} + m_e)$ since it assumed inside the fluid that $n_{ion} = n_e$. This assumption makes the charge density $\rho = (n_{ion}e - en_e) = 0$.

The equation which appears significantly different (of course it is nothing new) is :

$$\rho_m \left(\left(\frac{\partial \mathbf{u}}{\partial t} \right) + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\nabla p_{total} + \sum_{species} qn(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

$$\sum_{species} qn(\mathbf{E}) = (n_i e - n_e e)\mathbf{E} = n(e - e)\mathbf{E} = 0$$

and

$$\sum_{species} qn\mathbf{u} \times \mathbf{B} = (ne\mathbf{u}_{ion} - ne\mathbf{u}_{electron}) \times \mathbf{B} = \mathbf{J} \times \mathbf{B}$$

So the force equation becomes:

$\rho_m \left(\left(\frac{\partial \mathbf{u}}{\partial t} \right) + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\nabla p_{total} + \mathbf{J} \times \mathbf{B}$. Here ρ_m is the total mass density defined above and P is the pressure due to both ions and electrons.

Using Maxwell we can eliminate the current from the one fluid force equation

$\mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} = -\nabla \left(\frac{B^2}{2\mu_0} \right) + (\mathbf{B} \cdot \nabla) \frac{\mathbf{B}}{\mu_0}$ where we used the vector identity:

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

The one fluid equation becomes:

(I) $$\rho_m \left(\left(\frac{\partial \mathbf{u}}{\partial t} \right) + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\nabla \left(p + \left(\frac{B^2}{2\mu_0} \right) \right) + (\mathbf{B} \cdot \nabla) \frac{\mathbf{B}}{\mu_0}$$

Notice that we have eliminated J and E from everything so far so lets now do the same for the Electric field in Maxwell's equation:

$$\times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, as.well.as. \mathbf{J} = \sigma\left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right)$$

becomes:

$$-\frac{\partial \mathbf{B}}{\partial t} = \quad \times \left(\frac{\mathbf{J}}{\sigma}\right) - \quad \times \left(\mathbf{u} \times \mathbf{B}\right) = \frac{1}{\mu_0 \sigma}\left(\quad \times \left(\quad \times \mathbf{B}\right)\right) - \quad \times \left(\mathbf{u} \times \mathbf{B}\right)$$

Using the vector identity:

$$\left(\quad \times \left(\quad \times \mathbf{B}\right)\right) = \quad (\quad \cdot B) - \quad ^2 B$$ We arrive at

(II) $$\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\mu_0 \sigma}\left(\quad ^2 \mathbf{B}\right) + \quad \times \left(\mathbf{u} \times \mathbf{B}\right)$$

So these two and the equation of state are all we need (except for the ability to solve them!). remember the force equation is non-linear.

Now let us solve the equation for the wave we will be doing an experiment on

The **ION ACOUSTIC WAVE**

We will first do the math and then contrast this wave to its analogy in our world.. a sound wave.

First of all lets assume that there is no magnetic field in the plasma and the magnetic field of the wave is zero as well. This means that all the terms in equation II are zero and that

$X\mathbf{E} = 0$. Equation 1 the force equation for an ion becomes:

4) $\rho_m \left[\frac{\partial v_i}{\partial t} + (v_i \cdot \quad) v_i\right] = en\mathbf{E} - \quad p$ where v_i is the ion fluid velocity.

Using $\mathbf{E} = - \quad \Phi$ equation 4 becomes:

5) $\rho_m \left[\frac{\partial v_i}{\partial t} + (v_i \cdot \quad) v_i\right] = -en \quad \Phi - \quad p$

Now we must use an equation of state for the ions: $P = \gamma n K T_i$, where T_i is the ion temperature. Now 5 becomes:

6) $\rho_m \left[\frac{\partial v_i}{\partial t} + (v_i \cdot \quad) v_i\right] = -en \quad \Phi - \gamma K T_i \quad n$. Here we assume that there are no spatial variations in the ion Temperature. ρ_m = nM

This equation is non-linear. We have to linearize it to see what happens to small amplitude waves. This is the same thing we did before when we studied the Debye length. This means that the perturbation of the fluid velocity due to the wave is very small, and terms which go like v^2 are very, very small. Therefore: $(v \cdot \quad)v \quad 0$. We will also do a one dimensional problem and assume that everything changes in the x direction only.

Let us assume that n = n0+n_1 (the background density is slightly perturbed by the wave, and that the local changes in the potential Φ are due only to the wave.

Since we will look for wave solutions we search for solutions which go like cos(kx-ωt) or sin(kx-ωt). Here k = 2π/λ (wavenumber) and ω = 2πf (angular frequency). Since by DeMoivre's theorm: $e^{i\theta} = \cos\theta + i\sin\theta$, { let $\theta = kx - \omega t$ } we will use the exponential form and take the real or imaginary part to get sines or cosines. Since:

$$\frac{\partial(e^{i(kx-\omega t)})}{\partial t} = -i\omega e^{i(kx-\omega t)}$$

$$and.. \quad e^{i(kx-\omega t)} = \frac{\partial(e^{i(kx-\omega t)})}{\partial x} = ike^{i(kx-\omega t)}$$

Equation 6 now becomes:

7) $$-i\omega M n_0 v_i = -en_0 ik\Phi - \gamma KT_i ikn_1$$

We now must eliminate Φ from equation 7. When we studied the Debye problem we used the fact the electrons are much more mobile than the ions and control the potential. The electrons have a Maxwellian distribution function. Also to first order the electron and ion densities are equal. This means that:

$$n_e = n = n_0 e^{e\Phi/kT_e}$$

Just as we did in the Debye case we expand the above

$$n_o e^{(e\Phi/KT_e)} \approx n_0(1 + \frac{e\Phi}{kT_e} + Higher\_order\_terms) \quad n_0 + n_1$$

So now equation 7 becomes:

8) $$\omega M n_0 v_i = en_0 k \frac{n_1}{en_0} kT_e + \gamma KT_i kn_1$$

The last thing to do is eliminate n_1, the perturbed density , and get an equation for v_1 only. We can do this by using the continuity of mass equation (b). We will write it down, linearize it and Fourier analyze it in one step. Try it!

$i\omega n_1 = n_0 ikv_1$ putting n_1 from this into 8 gives us:

9) $$\omega M n_0 v_i = kn_0 \frac{kv_1}{\omega} KT_e + \gamma KT_i k\left(\frac{n_0 kv_1}{\omega}\right)$$

The v_i 's cancel. Multiplying this out we get a dispersion relation (a relationship between ω and k) :

$$\omega^2 = k^2 (KT_e + \gamma KT_i) / M$$

9) $$\frac{\omega}{k} = \sqrt{\frac{KT_e + \gamma KT_i}{M}} = c_s$$ c_s is the ion SOUND velocity.

Note the ion sound velocity depends upon the electron temperature and the ion temperature. In our lab plasma the ions are much colder than the electrons so we take $T_i = 0$ in the above.

Contrast this to an ordinary sound wave. Here $v_{sound} = \frac{\omega}{k} = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma RT}{M_w}}$

Where M_w is the molecular weight, T is the gas temperature and R the gas constant (which is related to K)

These look the same but a plasma sound wave is different. First of all since there is a Φ in the plasma, plasma sound has an electric field. Ordinary sound does not. Note that sound in air travels at about 300 m/sec. The sound speed in a plasma may be we-written as :

10) $$c_s = 9.79 X 10^5 \sqrt{\frac{\gamma T_e}{\mu}}$$ in cm/sec. Here μ = mass ion/mass proton, and T_e is in electron volts. For an Argon plasma μ = 40 and for $T_e = 6$, and taking γ = 1we get the sound speed to be 3.79 $X10^5$ cm/sec or 126 times the speed of sound in air.