LAPTAG

Lecture notes on Waves/Spectra Noise, Correlations and ….

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All signals (sound, light, waves, mathematical functions) can be broken down into Fundamental building blocks. These are called the elements of the spectra:

A prism creates a spectrum of colors from white light.

The spectrum has all the component colors or elements that the "white" light is made of.

The prism is a spectrum analyzer for light.

Light is a Wave

The wave is made up of electric and magnetic fields. In vacuum the equation for the propagation of light is a differential equation which looks like:

$$\frac{\partial^2 \vec{E}}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

The light is moving in the z direction at speed c

C = 186,000 mi/sec=3X10^8 meter/s

E is the electric field of the wave

The solution of this equation is very simple:

$$E = A\sin\left\{2\pi\left(\frac{z}{\lambda}-\frac{t}{T}\right)\right\}+\cos\left\{2\pi\left(\frac{z}{\lambda}-\frac{t}{T}\right)\right\}$$

With T the wave period, T=1/f and f the frequency

λ is the wavelength (how long from crest to crest) and the wave satisfies

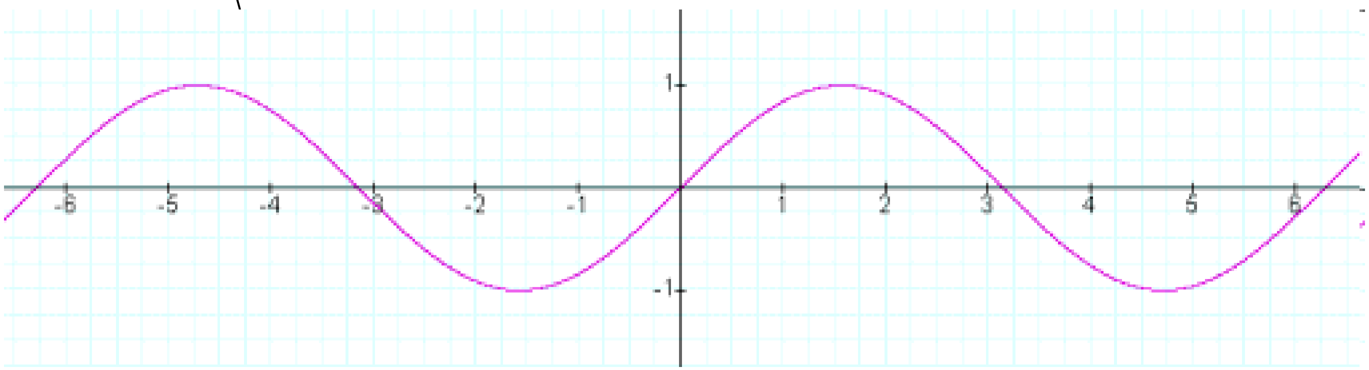
$\lambda f = c$ (This is called the dispersion relation. It relates the frequency and wavelength

To mathematically understand wave and spectra we have to go back to sines and cosines:

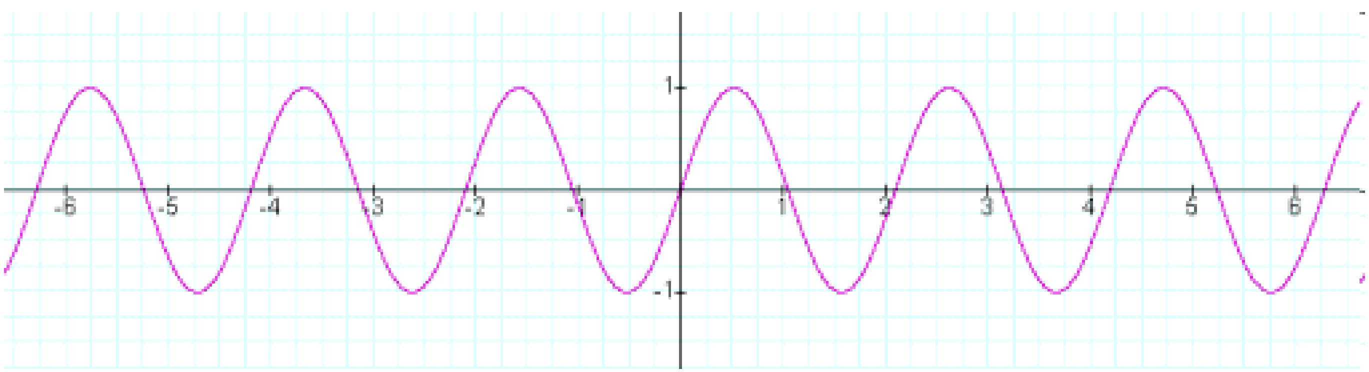
Review: y = sin(at) (the sine)

at is the “argument” of the sine. It is in radians

2π (radians) = 360 (degrees)



a=1

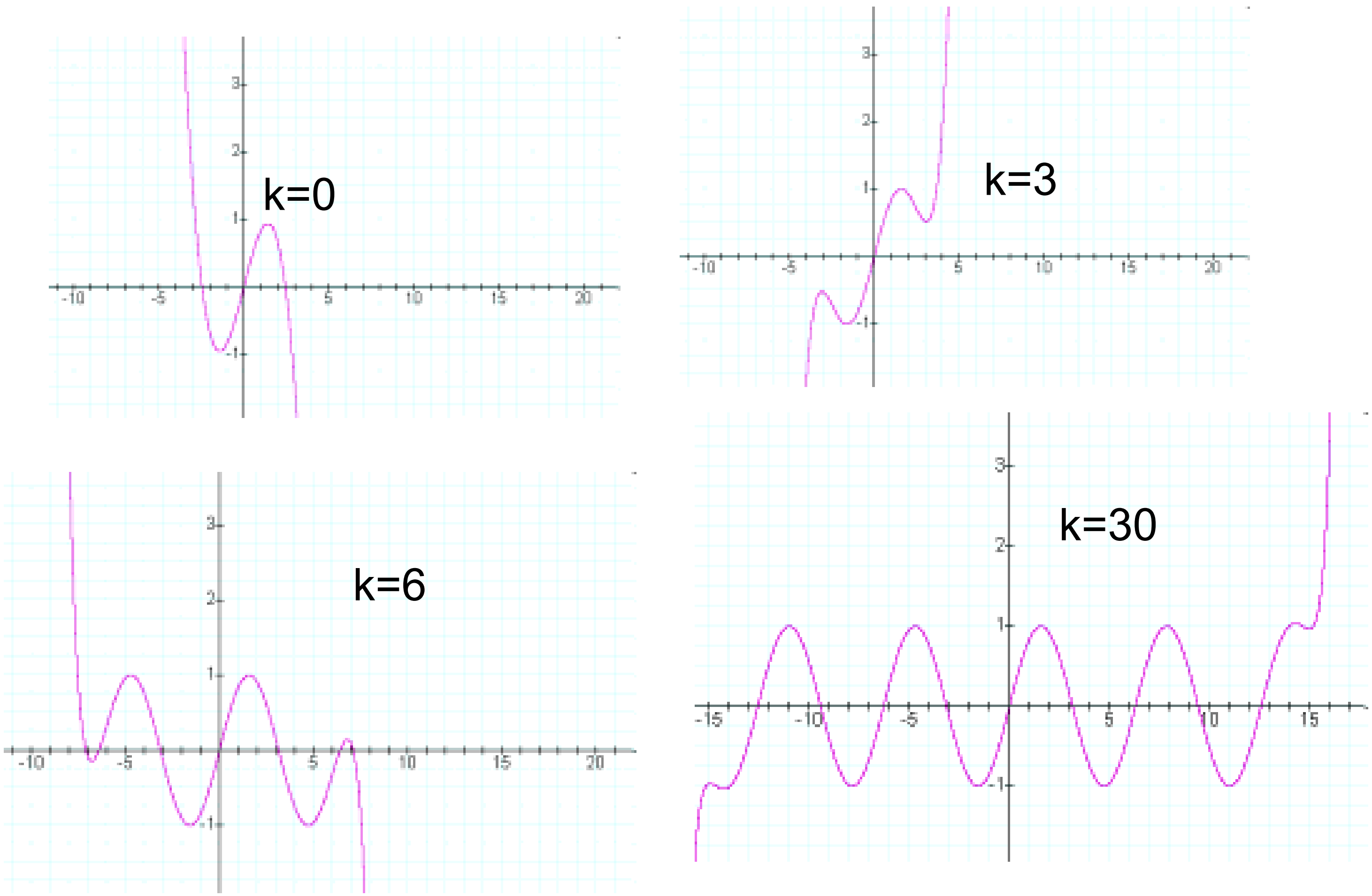


a=3

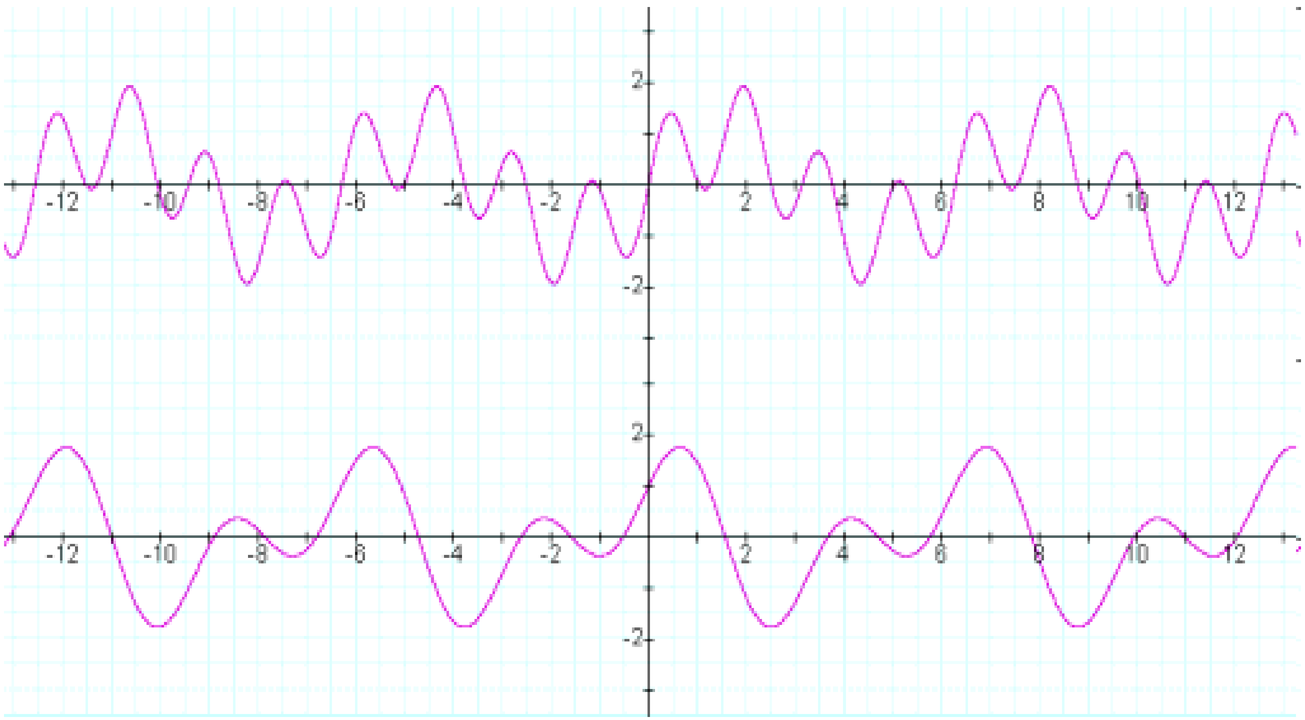
How long is the wavelength here?

The sine can also be expressed as an infinite series

sinx = Limit of y as k -> ∞ where y is: $$y = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

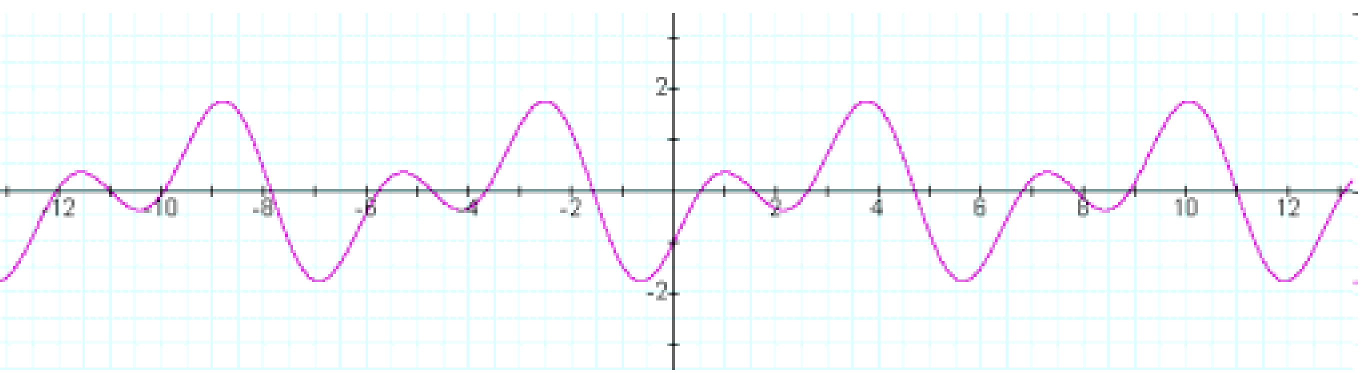


We can add sines and cosines up. Adding them is just like writing z= a+b



y = sinx+sin4x

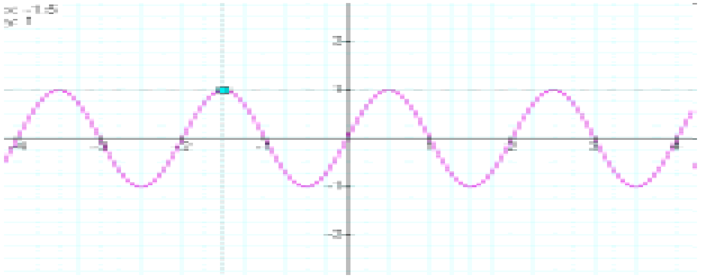
y = sin2x+cosx



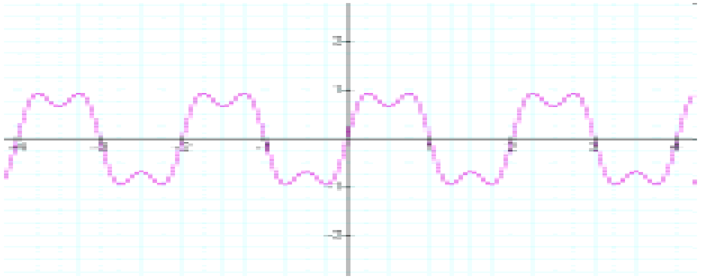
y=sin2x-cosx

$$y = \sum_{k=0}^{n} \frac{\sin(\,(2k+1)\,\pi x)}{2k+1}$$

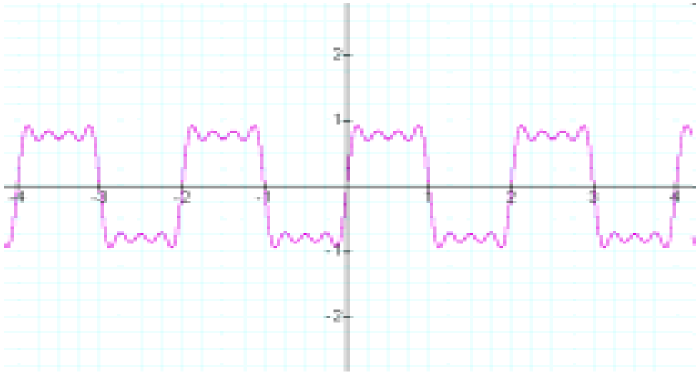
k=0 y1 = sin(πx)



k =1 ; y = y1 + sin(2πx)/3

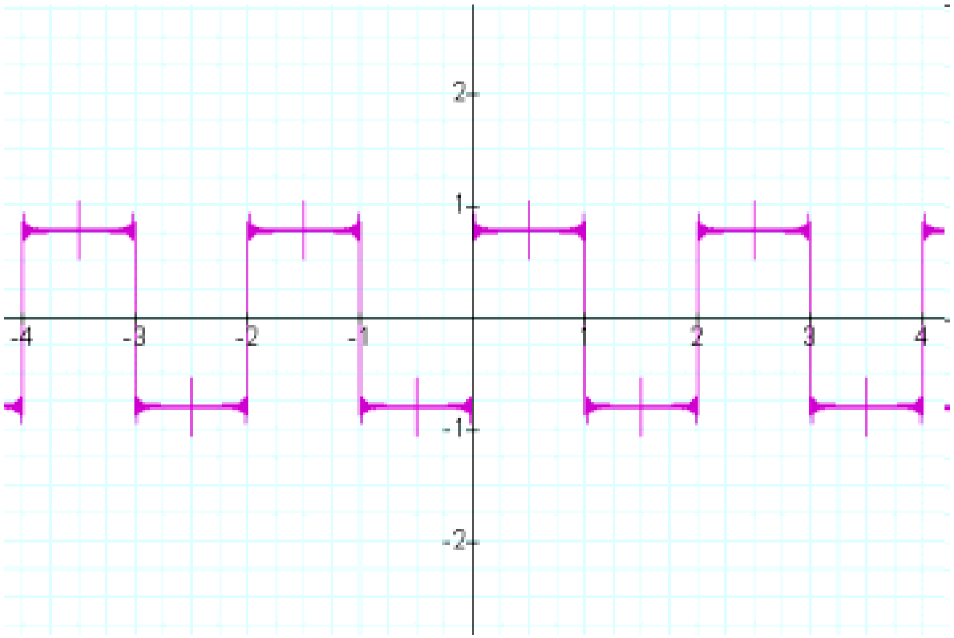


k = 3 ; y = y1 +y2+ sin(7πx)/7



$$y = \sum_{k=0}^{n} \frac{\sin((2k+1)\pi x)}{2k+1}$$

Lets sum up 50 terms!



By summing many sine terms the series converges to a shape that does not look like a sine function . In this case the summation is a square wave. Note that the wave repeats itself with a period of 4.0 units.

Mathematical functions (that are periodic) can all be expressed as an infinite sum of sines and cosines each multiplied by the right constant (like 1/3 or 1/7 as in the previous example)

If F(z) is any periodic function the it can be proven that :

$$F(z) = B_0 + \sum_{m=1}^{\infty} A_m \sin(mkz) + \sum_{m=1}^{\infty} B_m \cos(mkz)$$

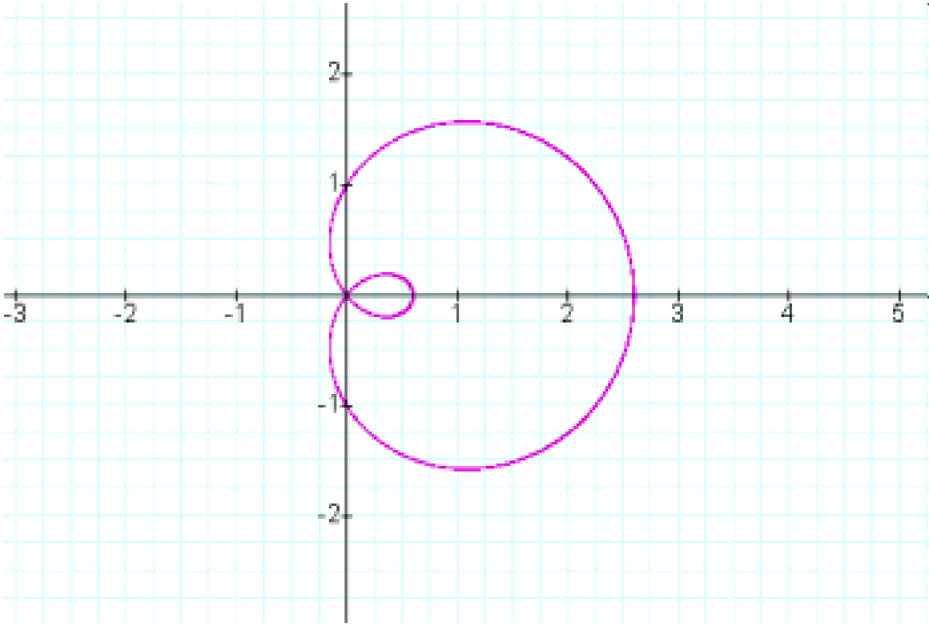
This is called a Fourier Series

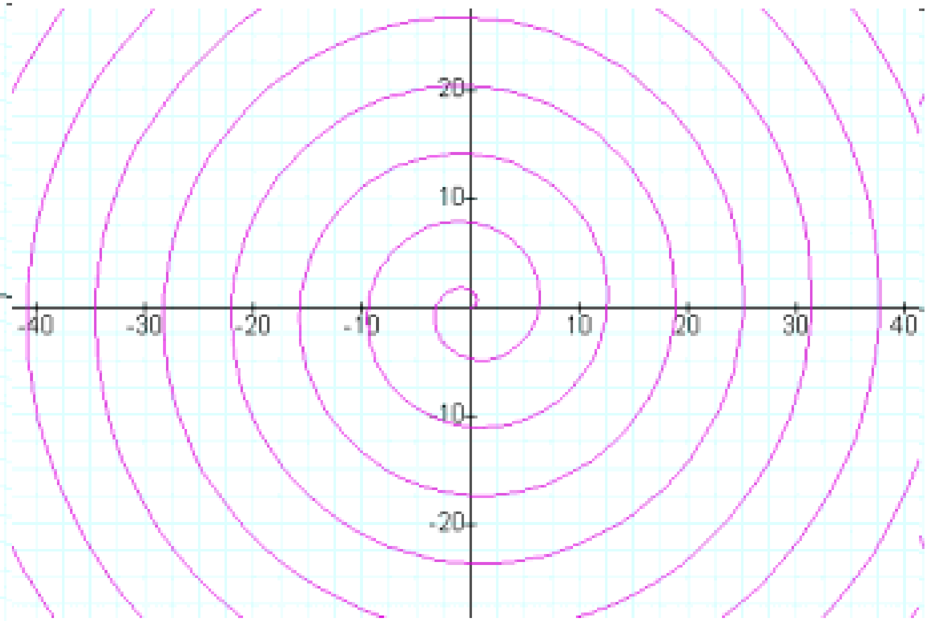
This is a function of z which a position like x or y. You can also have a series which describes a function F(t), function of time. The mathematics doesn’t care!

Some things can't be expressed as Fourier Series

This spiral is a function you cant use Fourier Series to describe

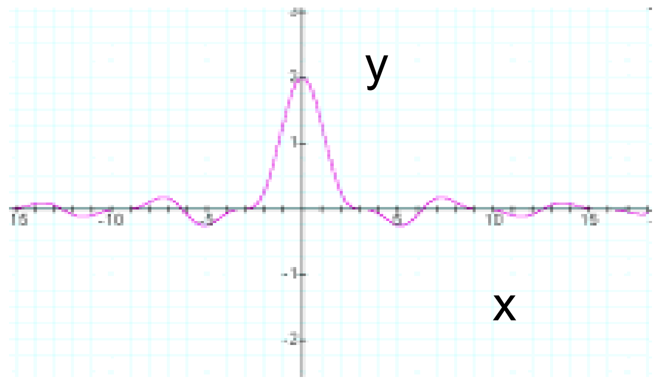
This is Pascal’s snail. No Fourier series here either!



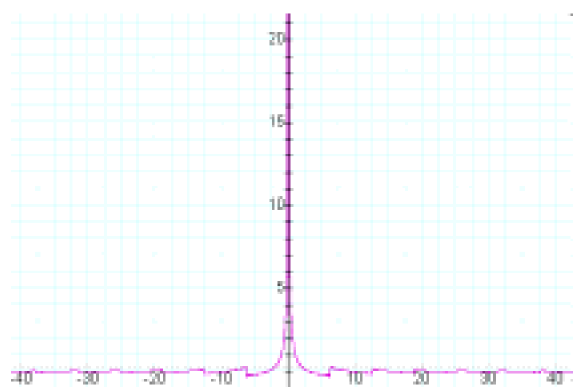


There are many other kinds of series, the ones we just showed are called Taylor series or power series. You can also construct series out of sines and cosines:

$$y = \left(\sum_{n=1}^{2} \left(\frac{1}{nx} \right) \sin(nx) \right)$$



$$y = \left(\sum_{n=1}^{30} \left(\frac{1}{nx} \right) \sin(nx) \right)$$



As n goes to infinity the series becomes a spike at $x = 0$

The mathematical form of the Fourier Series is:

$$F(z) = B_0 + \sum_{m=1}^{\infty} A_m \sin(mkz) + \sum_{m=1}^{\infty} B_m \cos(mkz)$$

B_0 is a constant

A_m and B_m are the m_{th} component of the spectrum of the wave. They are called the Fourier coefficients.

Note $k = 2\pi/\lambda$ where λ is the wavelength in the z direction.

What are the A's and B's?

$$B_0 = \frac{1}{\lambda} \int_{z}^{z+\lambda} F(z)dz$$

What do these symbols mean??

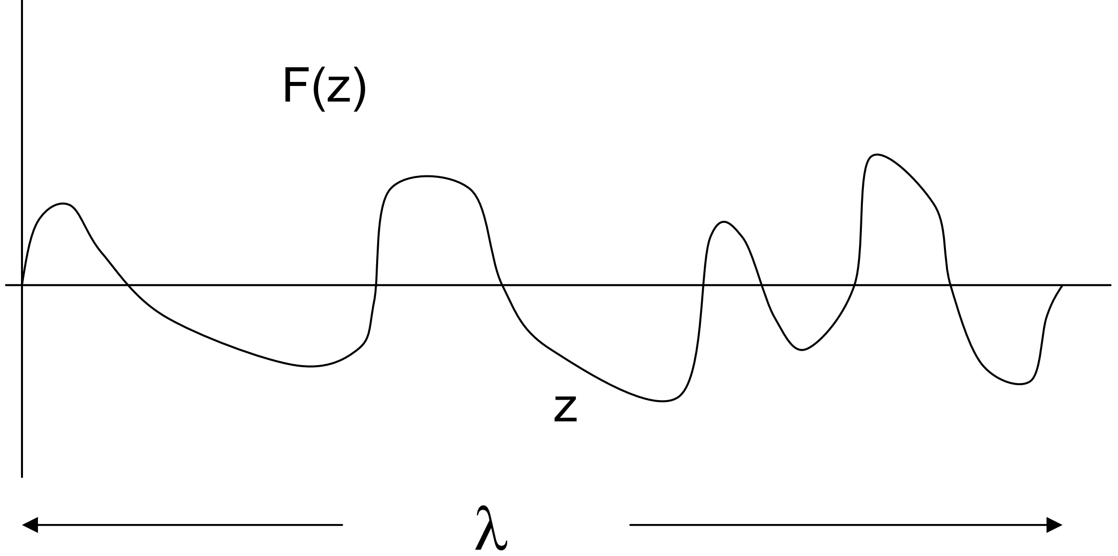
$$A_m = \frac{2}{\lambda} \int_{z}^{z+\lambda} F(z)\sin(mkz)dz$$

$$B_m = \frac{2}{\lambda} \int_{z}^{z+\lambda} F(z)\cos(mkz)dz$$

z is a spatial position, k is a constant. This is a series representing a function that changes in space. Time is fixed.

$$A_m = \frac{2}{\lambda}\int_{z}^{z+\lambda} F(z)\sin(mkz)\,dz$$

This function repeats itself every distance λ along the z axis.



What this "integral" means is for every position z within one wavelength take the function F(z) , which could be an experimental measurement, multiply by the sine:

sin(mkz)= sin(2πmz/λ)

Then go to the next z (which is a very small distance away), multiply by the next sine and add it to the first. This is the same as adding an infinite number of terms together.

The integral is really infinite sum

For example

$$A_m = \frac{2}{\lambda}\int_z^{z+\lambda} F(z)\sin(mkz)dz$$

$$A_m = \frac{2}{\lambda}\left[F(z_0)\sin(mkz_0) + F(z_1)\sin(mkz_1) + F(z_2)\sin(mkz_2) + ...F(z_n)\sin(mkz_n)\right]$$

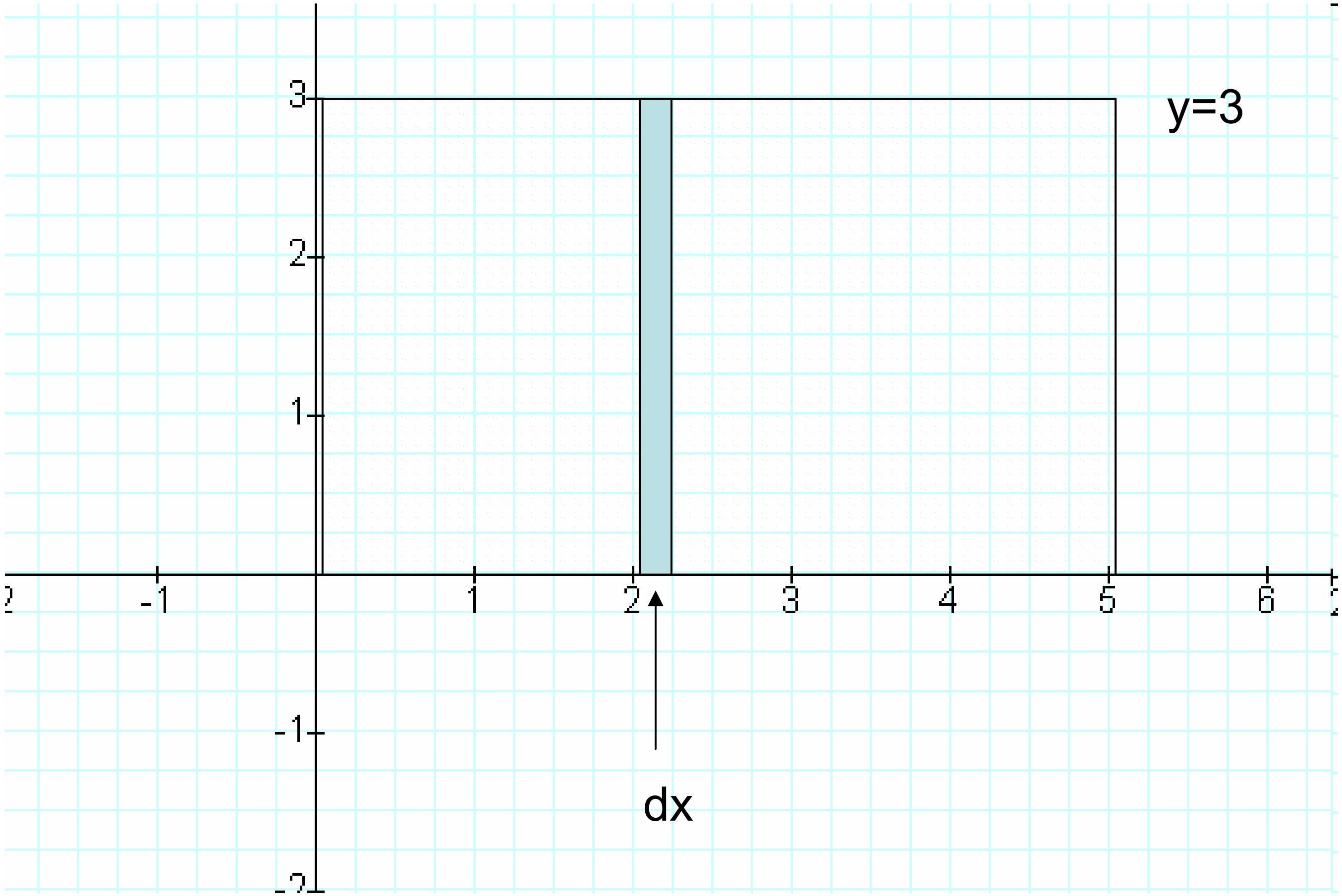
$$where\ z_1 - z_0 = \varepsilon \quad \text{and}\ \varepsilon \to 0$$

The infinite number of terms occurs because the difference in z positions at which you evaluate stuff is so small it is just about zero!

An integral is the area under

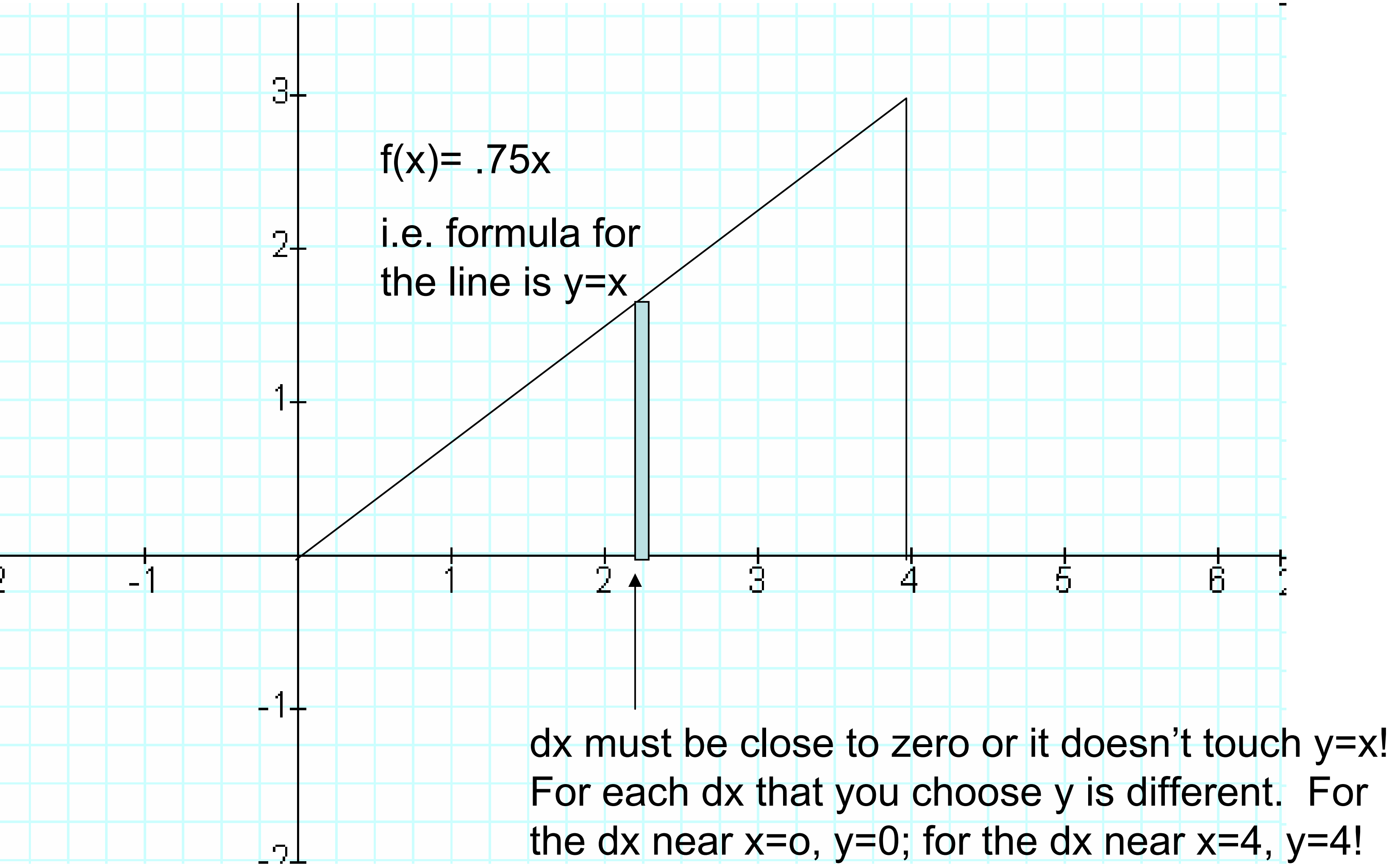
A curve. Here y=f(x)=3 (a constant!)

$$\int_{x=0}^{x=5} 3dx = 3x \Big|_0^5 = 3(5) - 3(0) = 15$$



The integral means take a dx (which is very) , multiply by the value of y above the

Dx, shift your dx and keep on doing it until you hit the limit for x.



$$\int_0^4 .75x\,dx = \frac{3}{4}\left(\frac{x^2}{2}\Big|_0^4\right) = \frac{3}{4}\left(\frac{16}{2} - 0\right) = 6 \quad \text{Area} = \frac{1}{2}(base \times height) = \frac{1}{2}(4 \times 3) = 6$$

The Fourier series for a function which changes in time is :

$$\mathrm{x}(\mathrm{t}) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega t + b_n \sin n\omega t\right) \qquad \omega = 2\pi \mathrm{f} = \frac{2\pi}{\mathrm{T}}$$

$$\mathrm{a_n} = \frac{2}{T}\int_0^T x(t)\cos n\omega t\,dt \qquad \mathrm{n} = 0,1,2,3...$$

$$\mathrm{b_n} = \frac{2}{T}\int_0^T x(t)\sin n\omega t\,dt \qquad \mathrm{n} = 1,2,3...$$

There is an alternate and equivalent way to write x(t)

$$x(t) = a_0 + \sum_{n=1}^{\infty} X_n \cos(n\omega t + \theta_n) \quad \theta_n \text{ is the phase of each term}$$

$$where\ \mathrm{X_n} = \sqrt{a_n^2 + b_n^2} \qquad \theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right) \quad \mathrm{n} = 1,2,3...$$

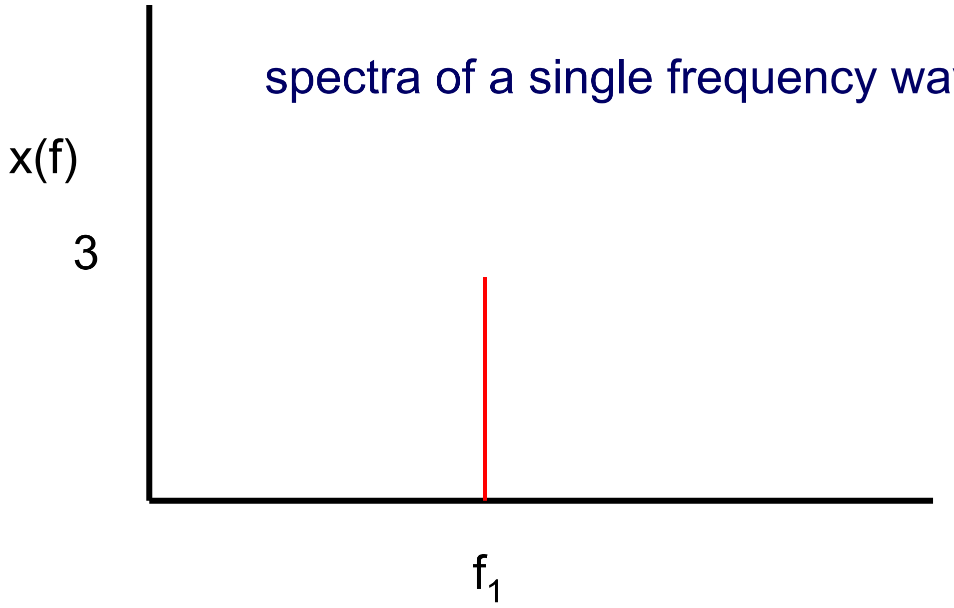
Now lets introduce the concept of spectra

WE know that any time series (such as our data) can be represented by a series:

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\omega t + b_n \sin n\omega t \right) \quad \omega = 2\pi f$$

The spectra of x(t) tells us how much of each frequency is present in the signal. Suppose the signal is composed of one sine wave only :

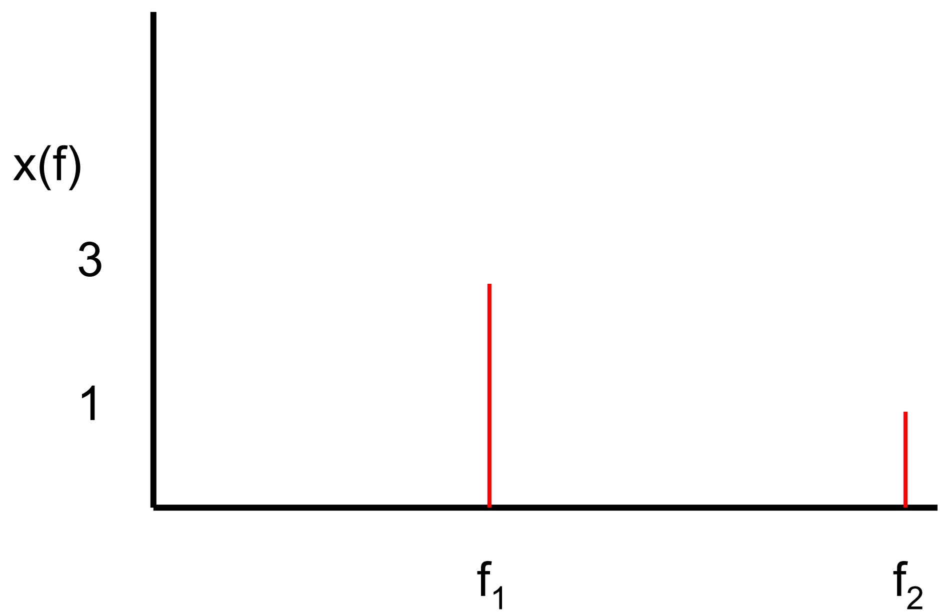
$x(t) = 3\sin(2\pi f_1 t)$



Suppose the signal is composed of two sine waves :

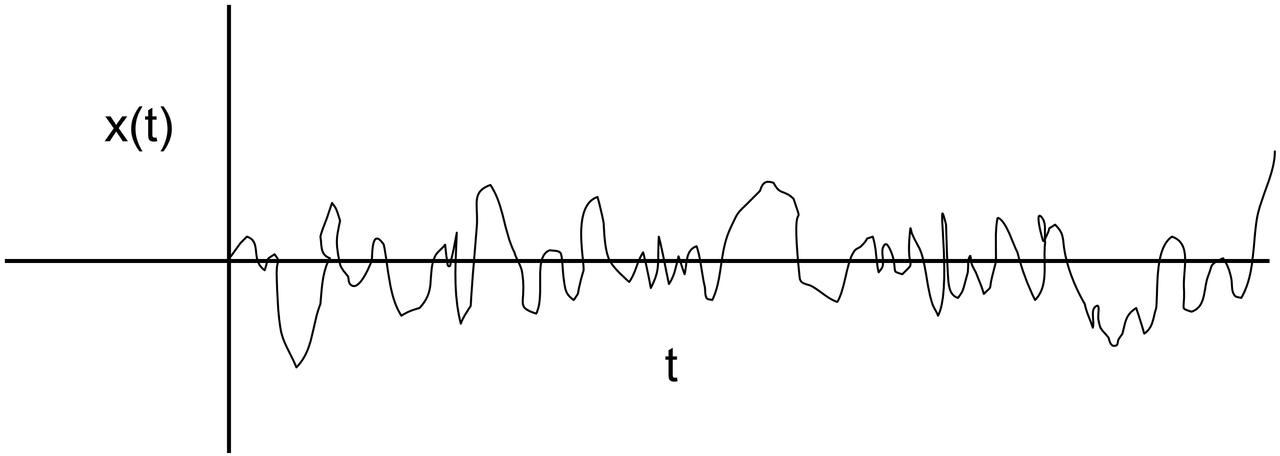
$x(t) = 3\sin(2\pi f_1 t)+\sin(2\pi f_2 t)$; where $f_2 = 3 f_1$

spectra of two frequency wave



Now there are two lines, one for each frequency

Now suppose that there are many frequencies in our waveform because there are many simultaneous waves in the media



Now the spectrum is continuous