LAPTAG

Lecture notes on Waves/Spectra Noise, Correlations and ….

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From the first lecture we discovered that any periodic mathematical function can be expressed as an infinite power series, a Fourier series:

$$\mathrm{x}(\mathrm{t}) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega t + b_n \sin n\omega t\right) \qquad \omega = 2\pi \mathrm{f} = \frac{2\pi}{\mathrm{T}}$$

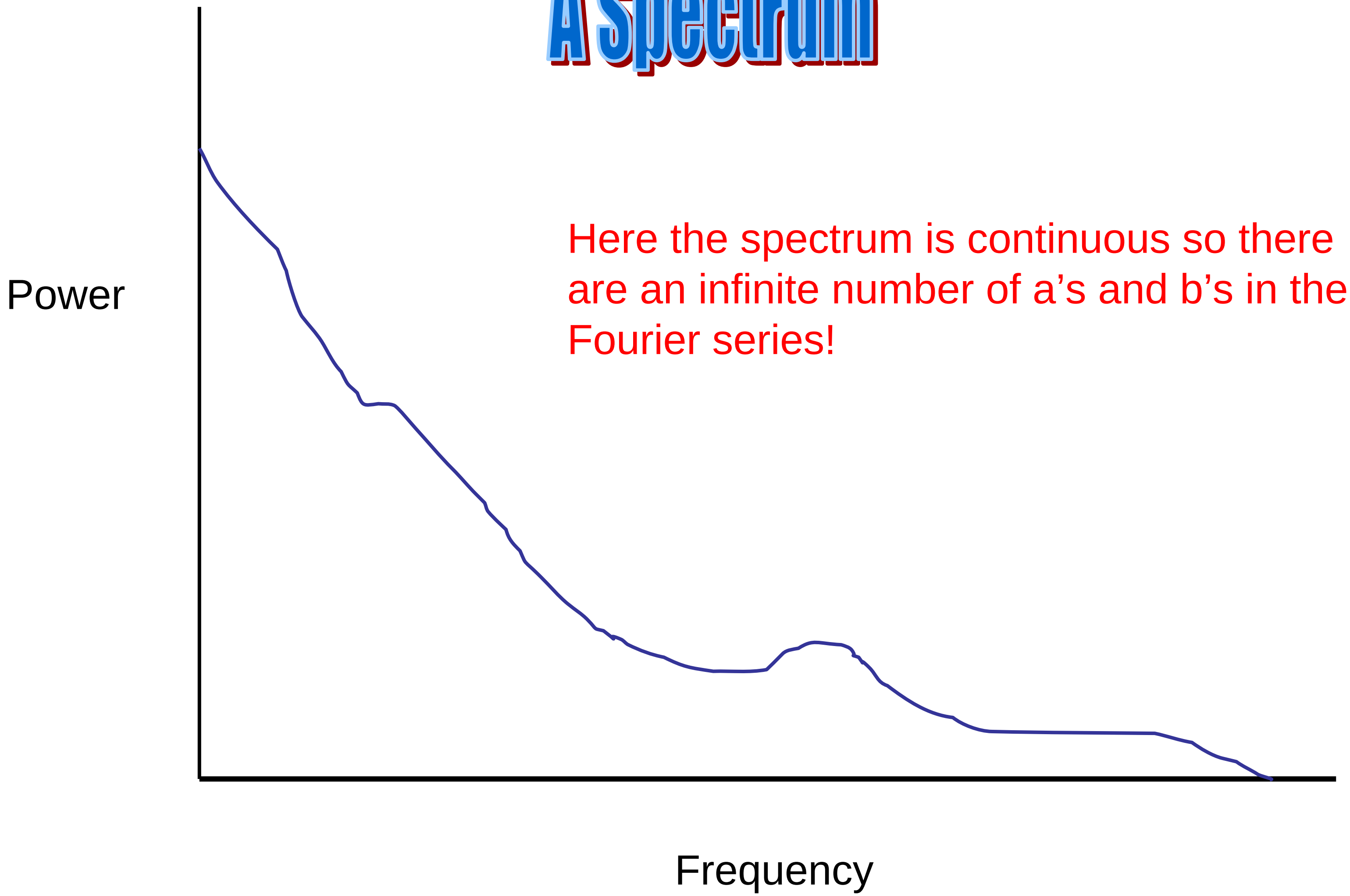
$$\mathrm{a_n} = \frac{2}{T}\int_0^T x(t)\cos n\omega t dt \qquad \mathrm{n} = 0,1,2,3...$$

$$\mathrm{b_n} = \frac{2}{T}\int_0^T x(t)\sin n\omega t dt \qquad \mathrm{n} = 1,2,3...$$

Our data is a function of t (time) and we collect it for T seconds. So we exploit the math by saying that our acquired data is one period long. *It repeats again and again in a mathematical space in Fourier land.*

Each a_n or b_n tells us "how much power" is in the n^{th} frequency. Therefore each a_n or b_n is a term in the spectrum.

A Spectrum

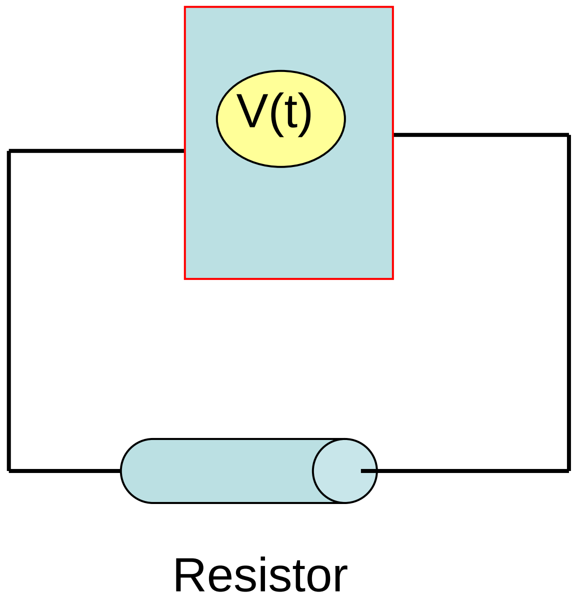


(color for light, pitch for sound)

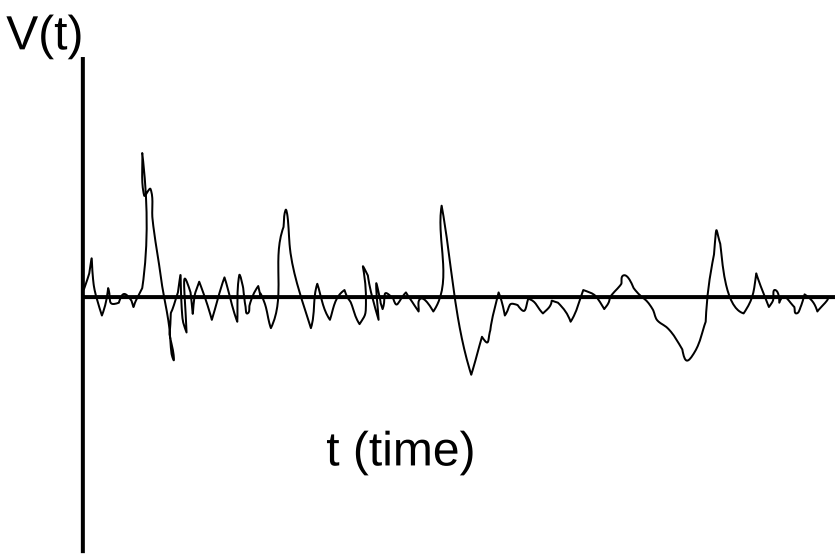
Questions for today:

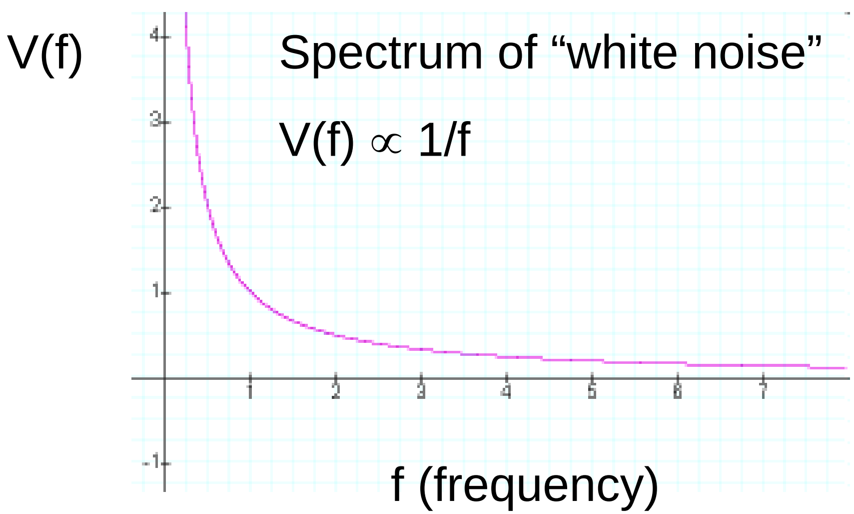
1) How do we find all the a’s and b’s from our data? (In other words how do we actually do a Fourier transform?)
2) What is phase?
3) What does phase have to do with Fourier Transforms ?
4) What is an inverse Fourier Transform?
5) Are there any pitfalls in using digitally acquired data?
6) Once we have a spectra are we all done or is there something else?

Some noise is truly random and is not made up of waves



Take a voltage and measure the voltage as a function of time across a resistor, V(t). Its average is zero because there is no battery in the circuit but:





1. How can we tell is what we have is random noise or if it is made out of waves?
2. If the noise is not random what is it made out of?
3. If our data has random noise in it and a valuable signal as well, how do we separate the signal from the noise?
4. What is the importance of the frequency spectrum?
5. How do we get the spectrum from x(t) ?
6. Is the spectrum enough to tell us about what the noise is composed of?
7. What are the techniques of extracting a valuable signal from an enormous background of noise (where the noise is a million times larger than the signal)?
8. How do you do all of these when your data is digital? Digital data is collected at discrete time steps (example one sample per microsecond). It is not continuous in time

These are complicated issues. Lets first concern ourselves with a signal that looks like random noise but is actually made up of many waves.

Some of the answer :

1) We get the spectrum from the Fourier transform

2) The noise may be made of waves so we must determine information about each of these waves. This information is contained in the dispersion relation. (relation between frequency and wavelength)

If the noise is made of waves they may damp out (get smaller), or grow as we move further away from the source of the waves.

Sometimes the waves can interact with each other as they travel away from their source. They could give “birth” to new waves, which would make the spectrum different from place to place.

How do we find the answer to question 2 ?

Much of the answer lies in what are called correlations.

The purpose of these lectures is to teach you what these correlations are and how to calculate them for data.

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To do this we have to learn some mathematical concepts.

a) What complex numbers are

b) How complex numbers are related to sines and cosines.

c) How complex numbers are related to the wave properties.

d) What Fast Fourier transforms (FFT) are.

e) How FFT's are related to correlation functions

f) How to use all this to analyze your data.

The use of phase is a key part in this analysis. To continue our analysis we have to introduce another mathematical entity:

Complex Numbers

This is all due to a famous identity due to DeMoivre:

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{where i} = \sqrt{-1}$$

Ok but what is e ?

and what are complex numbers ?

e ?

e is a transcendental number like π

e = 2.7182818284590452….   Or by an infinite series:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + + + \frac{1}{n!} \quad \text{, with } \frac{1}{3!} = \frac{1}{3 \bullet 2 \bullet 1} = 1/6$$

Properties of e:

$e^a e^b = e^{a+b}$ $\qquad$ $e^a / e^b = e^{a-b}$ $\qquad$ $(e^a)^b = e^{ab}$

multiplication and division become addition and subtraction !

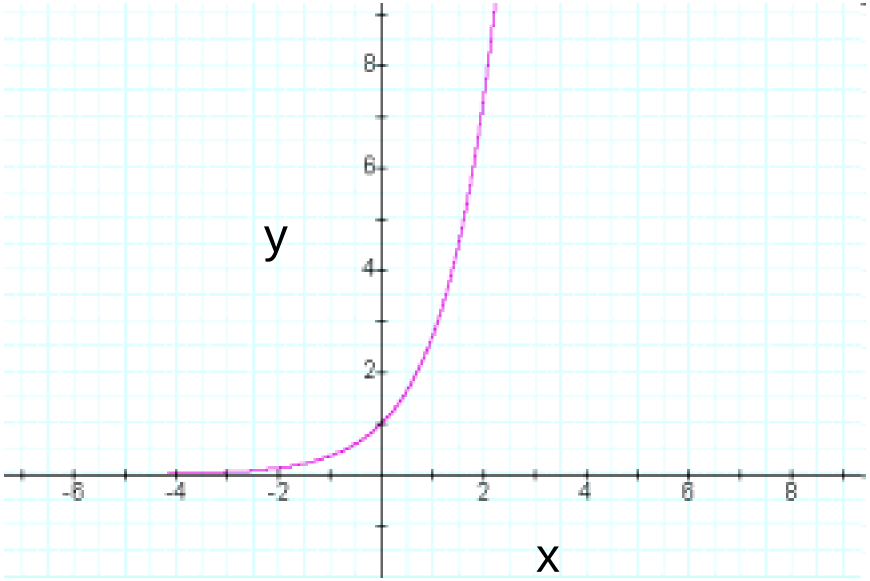
$y = e^x$; how do you find x if you know what y is?

This introduces us to a new mathematical function the natural logarithm.

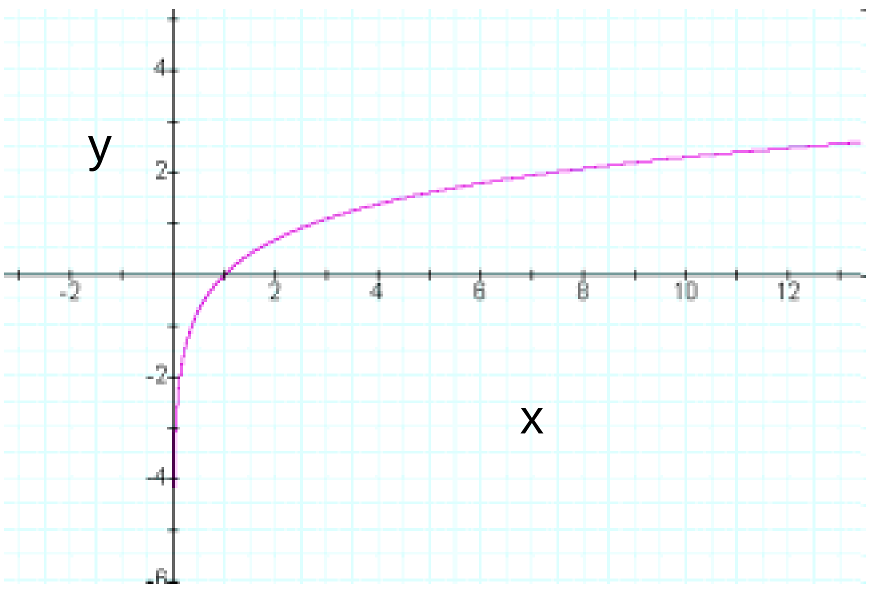
It has the property such that

$\ln(y) = x$

Graph of $y = e^x$



Graph of $y = \ln x$



So lets reiterate now that we know what e is:

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{where i} = \sqrt{\text{-1}}$$

e is related to sines and cosines as well as logs. Since sines and cosines are solutions to the wave equation so is e!

$$e^{-i\theta} = \cos\theta - i\sin\theta$$ We will see where this comes from soon

$$e^{i\theta} - e^{-i\theta} = \cos\theta + i\sin\theta - \cos\theta + i\sin\theta$$

$$Therefore: \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$Similarly: \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{where i} = \sqrt{-1}$$

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Any complex number may be written as:

$$\tilde{z} = a + ib \quad \text{The} \quad \sim \quad \text{means complex}$$

Thus it looks like complex numbers are just the right thing to use in Fourier Analysis

$$\text{also: } \tilde{z} = \text{ ze}^{\text{i}\theta}$$

$$z = \sqrt{a^2 + b^2} \quad \text{magnitude of X}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) \quad \text{phase}$$

So what's a complex or imaginary number??

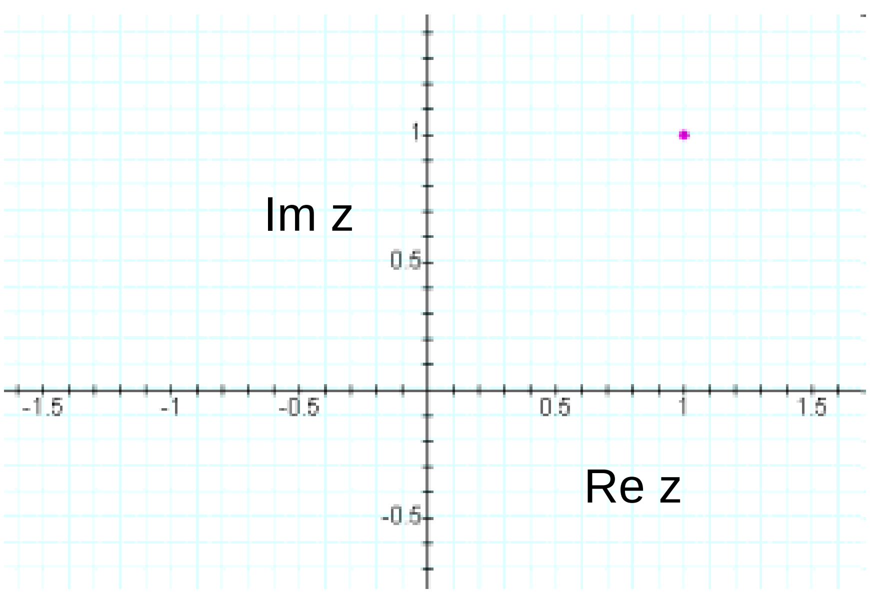
• Many years ago mathematicians had a big problem with the concept of zero

• After that they had a big problem with the concept of negative numbers

• After that they had a big problem with the concept of infinity

• After that they had a big problem with the concept of transcendental numbers such as π (π = 3.141592654…..and on forever)

• Now You are having a big problem with complex numbers!

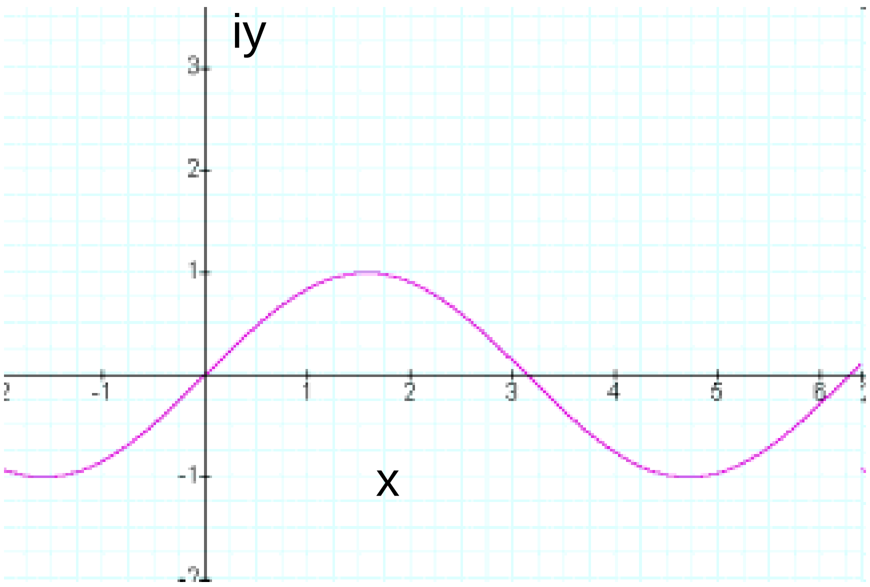


They have a real part and an imaginary part (the part multiplied by i) therefore you can draw one complex number on two axis

example: z=1+i

Review :A complex number $z=x+iy=Ae^{i\theta} = A(\cos\theta+i\sin\theta)$

These are all equivalent representations. What do they look like?



Example

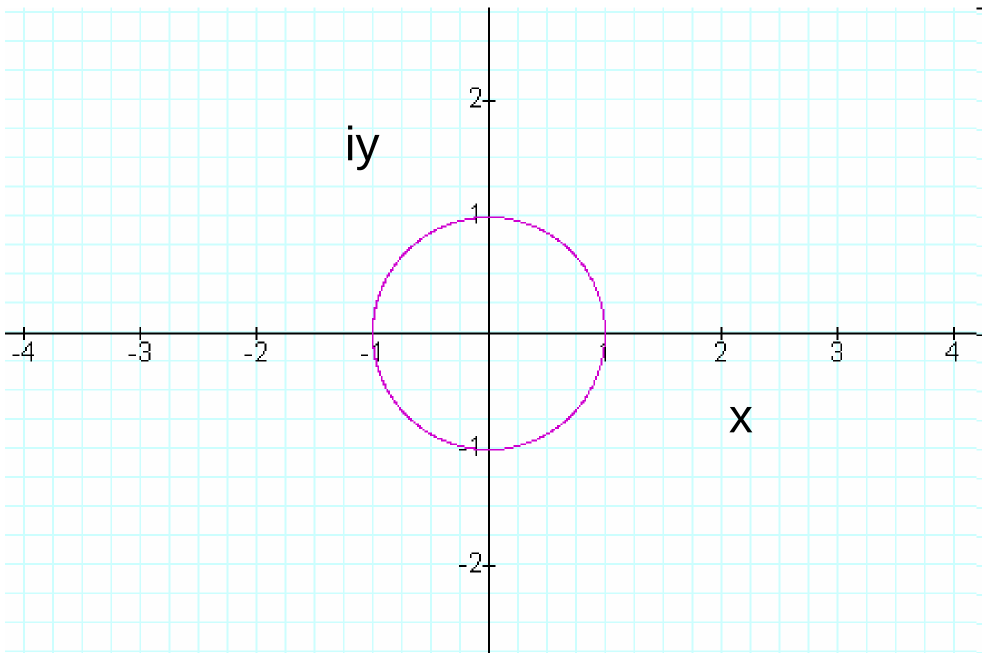
$z= \mathrm{Im}(e^{ix})$

$z= \mathrm{Im}(\cos x+i\sin x)=\sin x$

What would $z=e^{i2\pi t}$ look like?

This is the equation for as circle!!!

$\check{z} = e^{2\pi i t} = \cos(2\pi t) + i\sin(2\pi t) = x + iy$



When t=0 x=1, y=0, ($\cos\theta = 1$, $\sin\theta = 0$) ,When t =0.5 x= -1, y =0 ,

When t = 0.1 $\cos(.2\theta) = 0.809$, $\sin(.2\theta) = 0.588$; $(.809)^2 + (.588)^2 = 1$

How do you multiply two complex numbers?

Normally if a and b and real numbers then ab=ba if a = b then you get b^2

ž is a complex number then z^2 = ž ž* = (x+iy)(x-iy) = $x^2+y^2+ixy-ixy=x^2+y^2$

Here ž* is called the complex conjugate of z

To take the complex conjugate replace i by –i whenever you see it

That's why in the previous example we get the equation
of a circle from ž =$e^{i2\pi t}$

Another example what does $x + iy = e^{2\pi iz}$ Look like?

$$\left(x+iy\right)\left(x-iy\right) = e^{2\pi iz}e^{-2\pi iz} = 1$$

$$x^2 + y^2 + ixy - ixy = x^2 + y^2 = 1 \quad \text{for any value of z}$$

$$x + iy = e^{2\pi iz} = \cos\left(2\pi z\right) + i\sin\left(2\pi z\right)$$

Equate the real and imaginary parts (they are two seperate axis)

$$x = \cos\left(2\pi z\right) \quad \text{y}=\sin\left(2\pi z\right)$$