LAPTAG

Lecture notes on Waves/Spectra Noise, Correlations and ….

W. Gekelman

Lecture 3, January 31,2004

From the first lecture we discovered that any periodic mathematical function can be expressed as an infinite power series, a Fourier series:

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega t + b_n \sin n\omega t\right) \qquad \omega = 2\pi f = \frac{2\pi}{T}$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega t\,dt \quad n = 0,1,2,3...$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\omega t\,dt \quad n = 1,2,3...$$

Our data is a function of t (time) and we collect it for T seconds. So we exploit the math by saying that our acquired data is one period long. *It repeats again and again in a mathematical space in Fourier land.*

Each a_n or b_n tells us “how much power” is in the n^{th} frequency. Therefore each a_n or b_n is a term in the spectrum.

So lets reiterate now that we know what e is:

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{where i} = \sqrt{-1}$$

e is related to sines and cosines as well as logs. Since sines and cosines are solutions to the wave equation so is e!

$$e^{-i\theta} = \cos\theta - i\sin\theta$$ We will see where this comes from soon

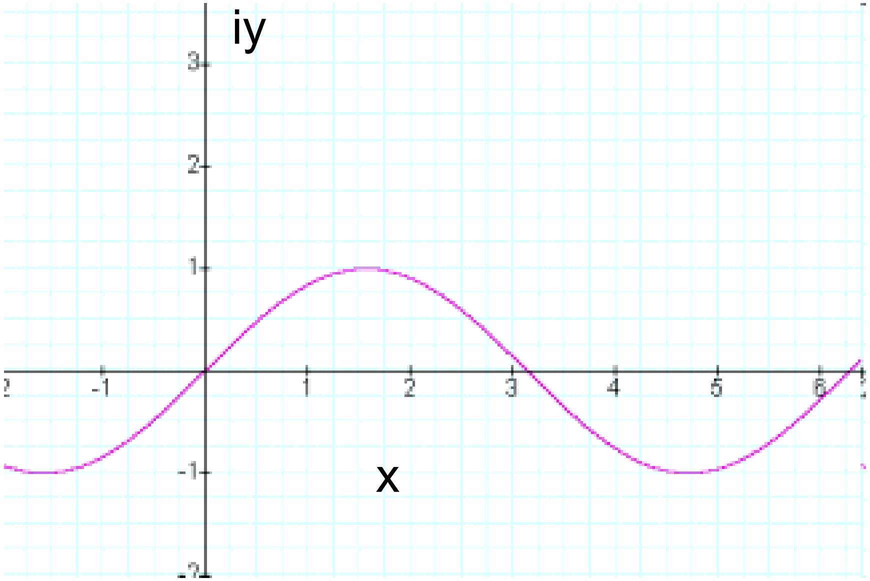
$$e^{i\theta} - e^{-i\theta} = \cos\theta + i\sin\theta - \cos\theta + i\sin\theta$$

$$Therefore: \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$Similarly: \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Review :A complex number $z=x+iy=Ae^{i\theta} = A(\cos\theta + i\sin\theta)$

These are all equivalent representations. What do they look like?



Example

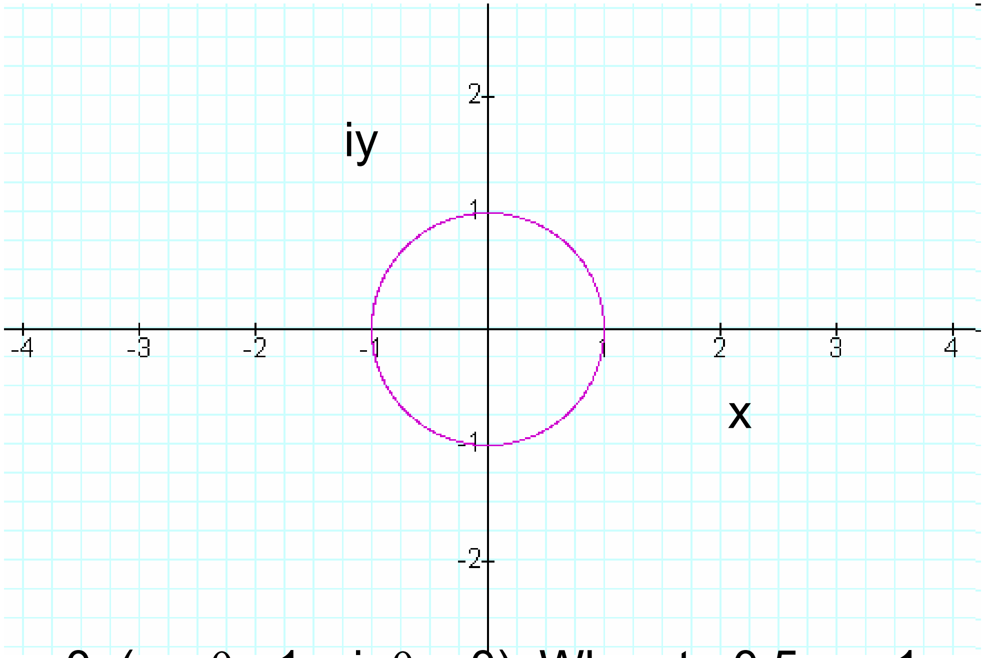
$z= \text{Im}(e^{ix})$

$z= \text{Im}(\cos x+i\sin x)=\sin x$

What would $z=e^{i2\pi t}$ look like?

This is the equation for as circle!!!

$\check{z} = e^{2\pi i t} = \cos(2\pi t) + i\sin(2\pi t) = x + iy$

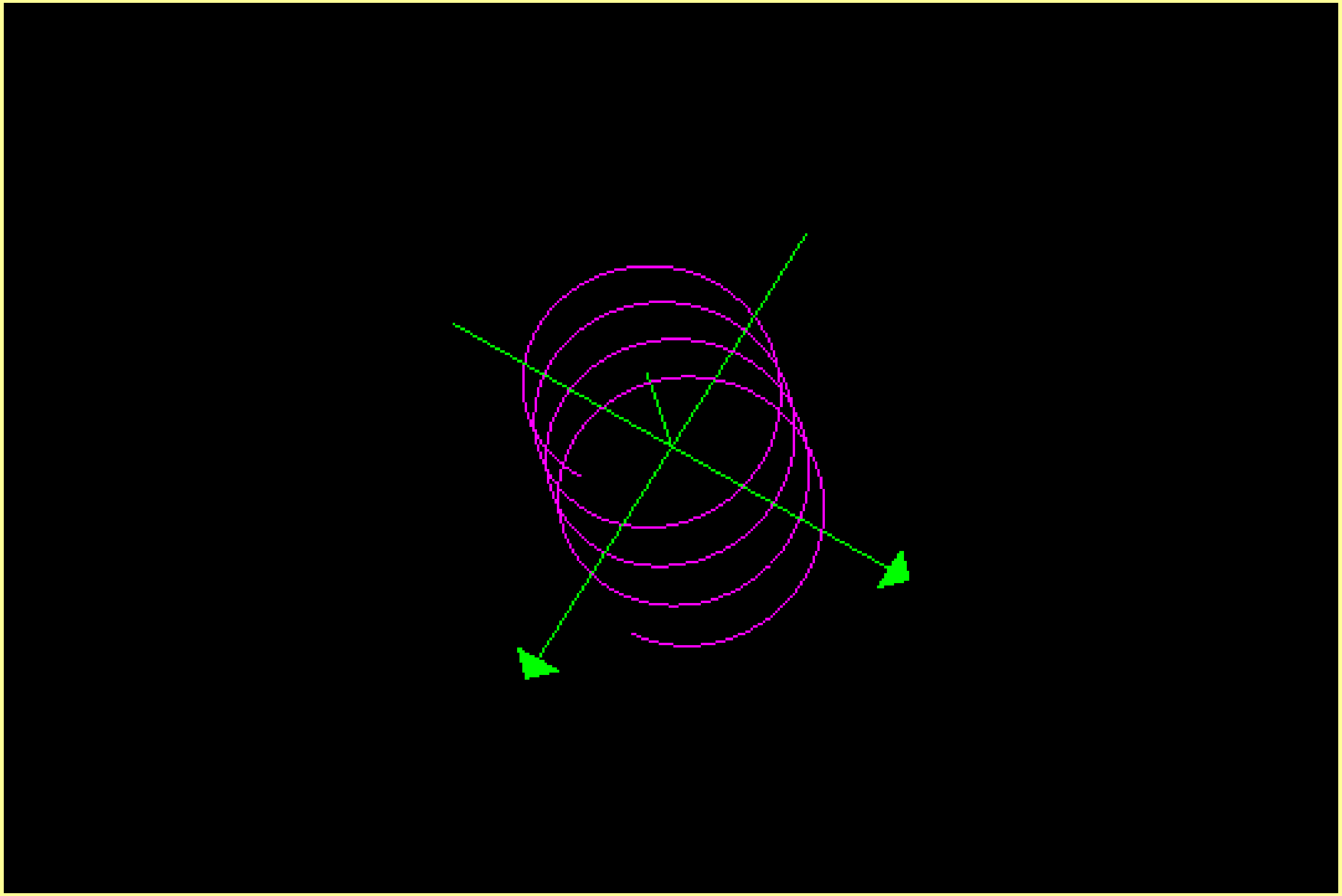


When t=0 x=1, y=0, ($\cos\theta = 1$, $\sin\theta = 0$) ,When t =0.5 x= -1, y =0 ,

When t = 0.1 $\cos(.2\theta) = 0.809$, $\sin(.2\theta) = 0.588$; $(.809)^2 + (.588)^2 = 1$

$$x + iy = e^{2\pi i z}$$

It's a spiral (in the picture the x and y axis have the arrows)



So here's what we are going to do:

- We will use infinite series to represent all mathematical functions

(note your data is a function x(t), you just happen to measure it!)

- Instead of saying that the series are sums of sines and cosines we will represent them as series of $A_n \sin(2\pi nt+\Phi_n)$
- Sometimes instead of using $A_n \sin(2\pi nt+\Phi_n)$

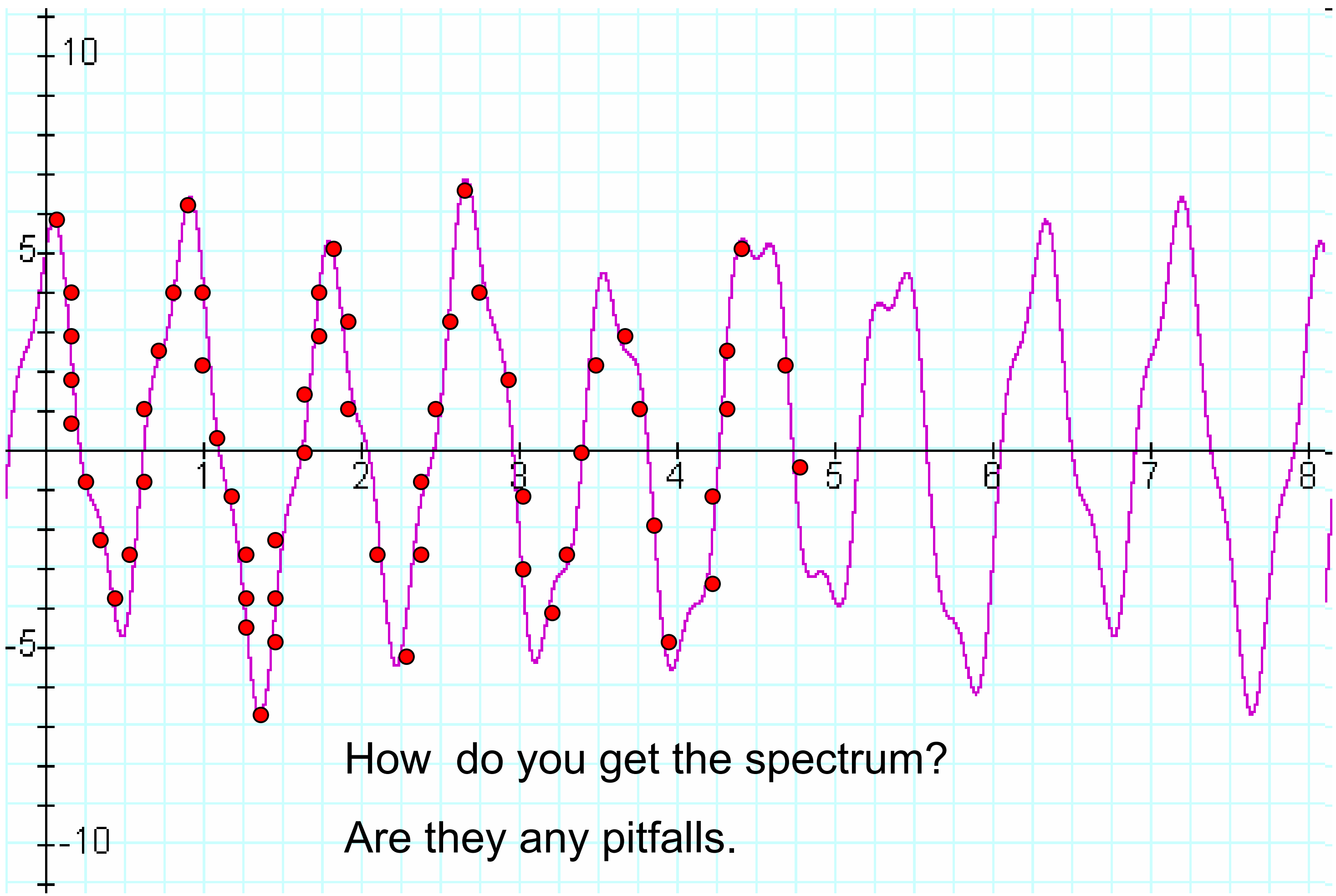
we will use complex numbers and write each term as

$$x_n(t) = A_n e^{i(2\pi nt+\Phi n)}$$

Note $x_n(t)$ is the n^{th} sine term in the time series , not the n^{th} time step!!!!

You will see how we do this as we go on to analyze the data

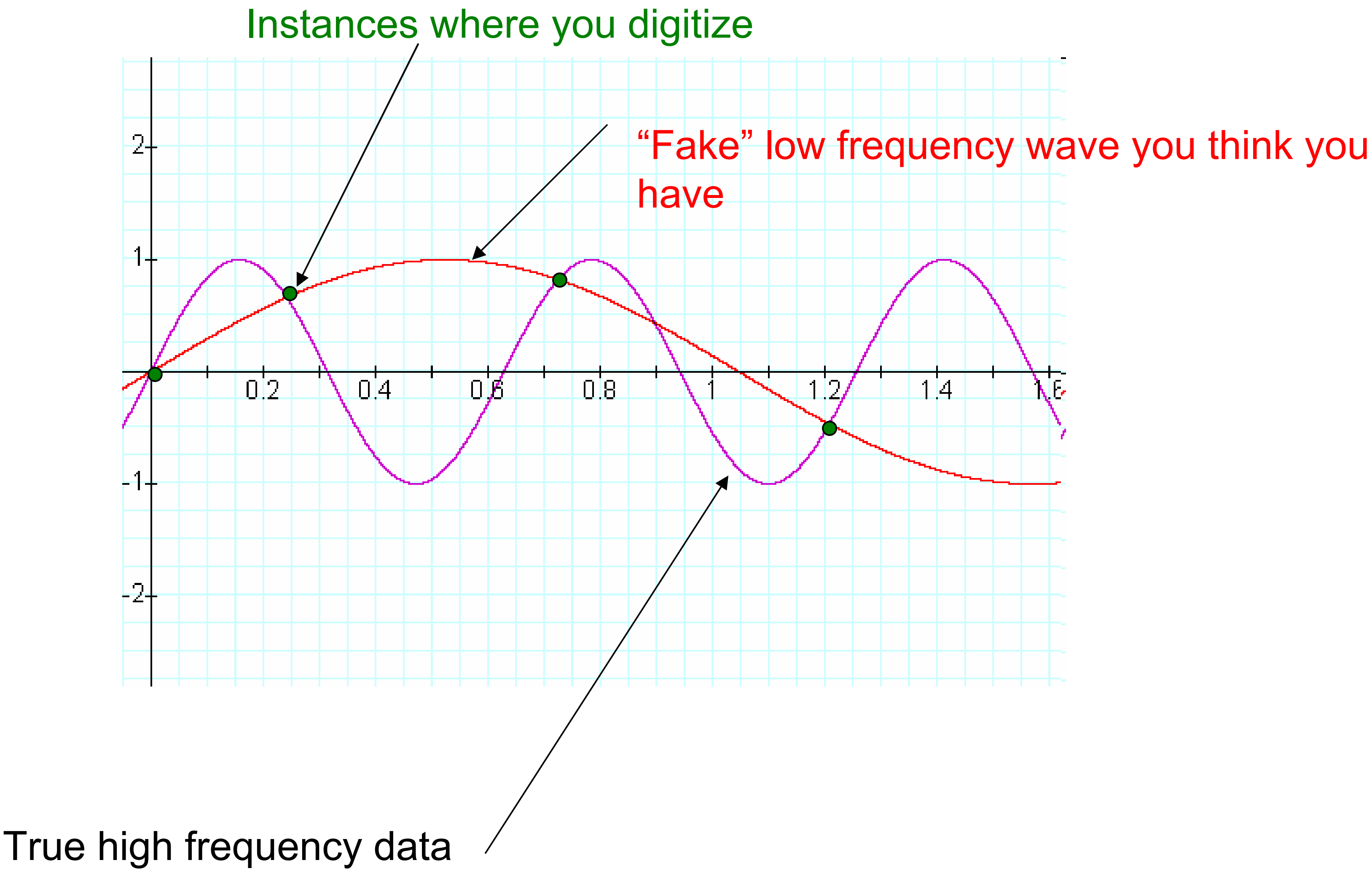
The purple continuous signal is the actual waveform but you digitize it at discrete times where the red dots are.



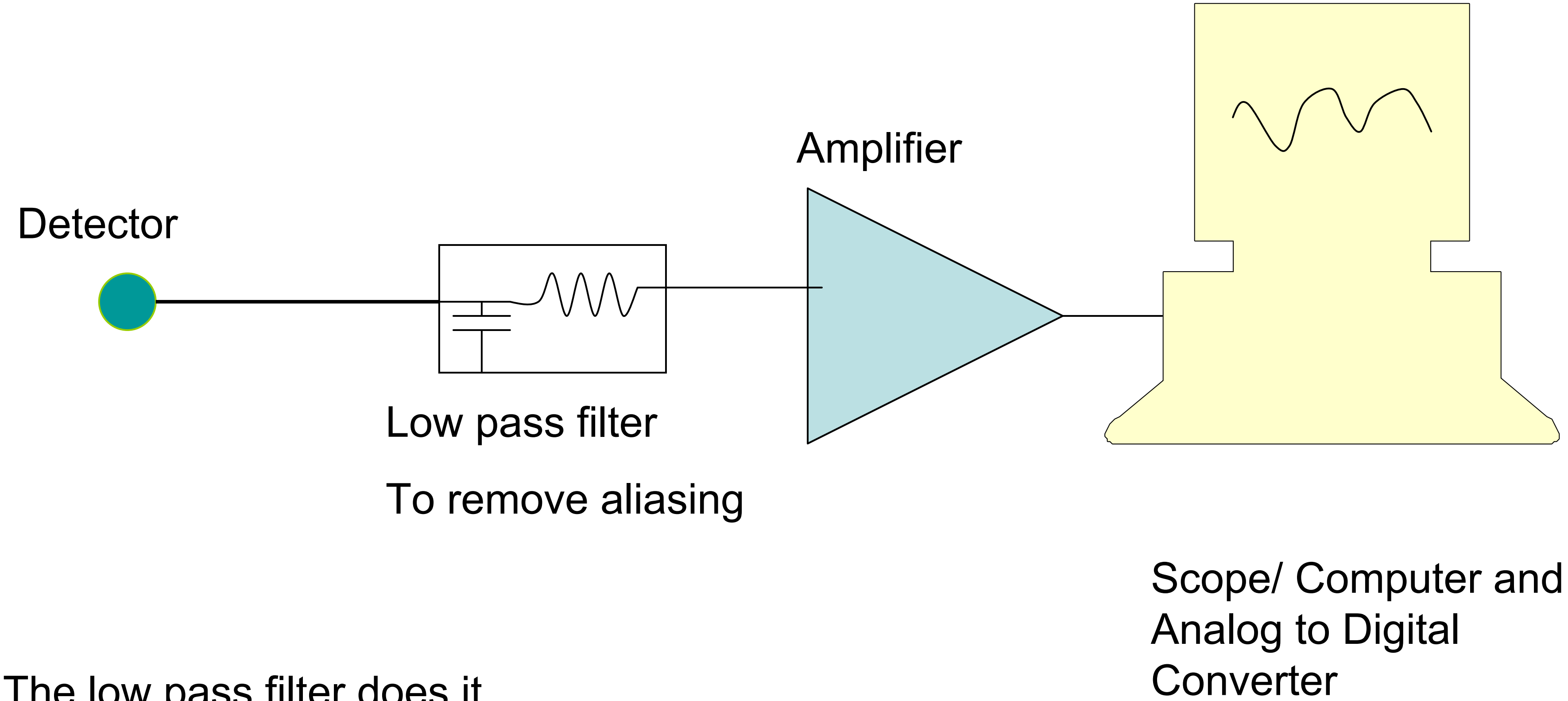
How do you get the spectrum?

Are they any pitfalls.

The biggest pitfall is called aliasing. Suppose you have a signal varying very rapidly in time and your digitizers have a time response which is too slow:



How do you get rid of aliasing?



The low pass filter does it.

Sometimes amplifiers have low pass filters.

Your digital data x(t) is a series to begin with since data is acquired at $t_1 = t_0 + dt$, $t_2 = t_1 + dt$, $t_3 = t_2 + dt$ …etc

Suppose you have N time samples in your data (the digitizer can store 1024 numbers). Suppose the digitizer takes data every microsecond ($dt = 1\mu s$).

The total length in time of your record is $T=Ndt = 1024*10^{-6} = 1.024X10^{-3}$ sec = 1.023 ms. $\approx$ 1 ms.

This is called T_p or the Nyquist frequency

What is the lowest frequency you can measure in your data?

Answer $f_{low} = 1/(2dt) = 1/2X10^{-3} = 5X10^2$ sec = 500 Hz

What is the highest frequency you can measure?

Ans: If you sample at 1 MHz then

Our digital data is a function of time x(t) and can be represented as:

$$\mathrm{x}(\mathrm{t}) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega t + b_n \sin n\omega t\right) \quad \omega = 2\pi \mathrm{f}$$

$$\mathrm{a_n} = \frac{2}{T}\int_0^T x(t)\cos n\omega t dt \quad \mathrm{n = 0,1,2,3...}$$

$$\mathrm{b_n} = \frac{2}{T}\int_0^T x(t)\sin n\omega t dt \quad \mathrm{n = 1,2,3...}$$

But since time is not continuous after we digitize t=m∆t, ∆t are the timesteps we digitize at:

$$\mathrm{x_m} \equiv \mathrm{x}(\mathrm{m}\Delta\mathrm{t}) = a_0 + \sum_{m=1}^{\frac{N}{2}}\left(a_n \frac{\cos n\omega m}{N} + b_n \frac{\sin n\omega m}{N}\right) \quad \omega = 2\pi \mathrm{f}$$

Note the time has been replaced by m/N and the limit does not go to infinity but to N/2 (the longest period we can measure due to the Nyquist limit!).

The weights a_n and b_n are also discrete and the formulas:

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega t dt \quad n = 0,1,2,3...$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\omega t dt \quad n = 1,2,3...$$

Become:

$$a_n = \frac{2}{N}\sum_{m=1}^{N} x_m \cos\frac{n\omega m}{N}$$

$$b_n = \frac{2}{N}\sum_{m=1}^{N} x_m \sin\frac{n\omega m}{N}$$

$$a_0 = \frac{1}{N}\sum_{m=1}^{N} x_m = < x > = 0$$

For a_0 we assume that the signals are made of waves and the average value for all waves is zero (<x> is the average value of x)

How many mathematical operations does it take to find all the a's and b's ???. We have collected N time steps of data

1. For every n and m evaluate $\theta=2\pi nm/N$
2. Find $\sin\theta$ and $\cos\theta$
3. Compute $x_n\sin\theta$ and $y_n\cos\theta$
4. Sum all the terms m=1,2,3….N/2
5. Go to next value of n and start again

This procedure takes approximately N^2 multiply add operations

If N = 1024 then 1,048,576. operations are required

Since finding the spectra is very important mathematicians have found a way to rapidly perform these mathematical operations and get the same result. This is called the:

Fast Fourier Transform (FFT)

How this is implemented is complicated and you may learn it one day in an advanced math course in college but the bottom line is the FFT requires

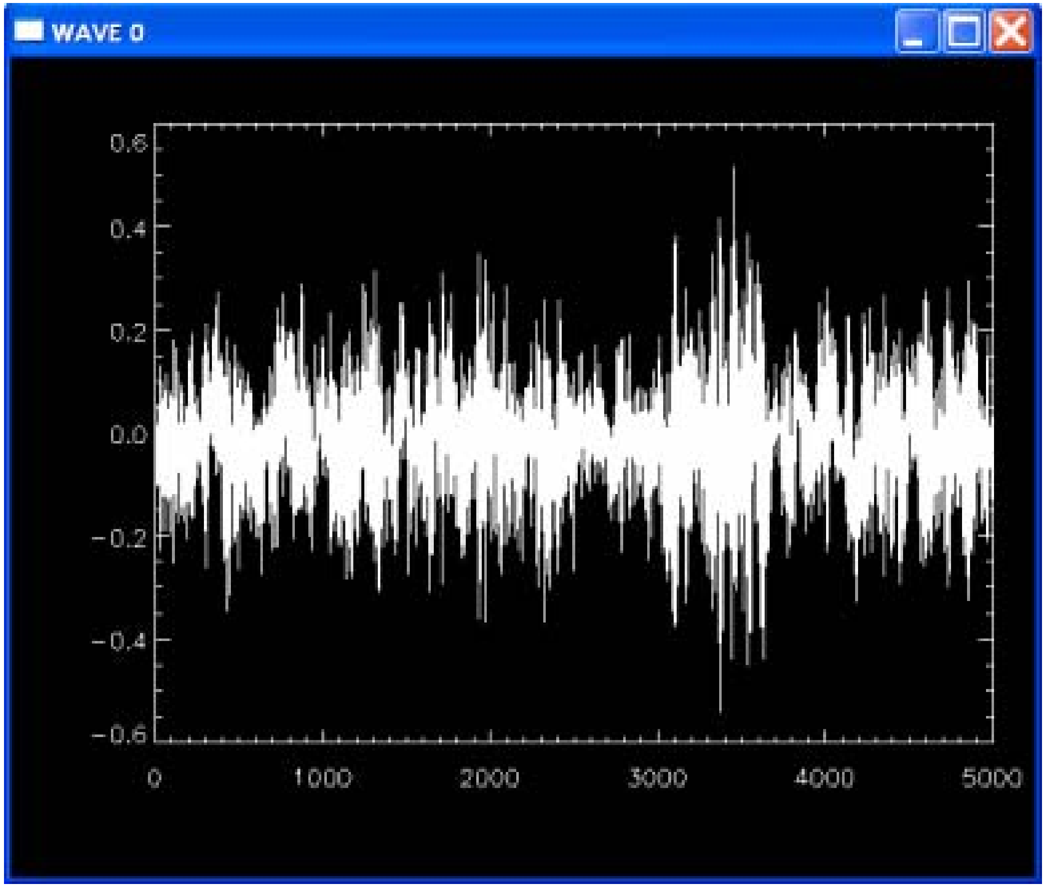
number of operations = $N/2p$ for data sets where the number of time steps is a power of 2 namely $N=2^p$.

Since $2^{10} = 1024$, number of operations = $1024/(2*10)=52$

This is a big win!

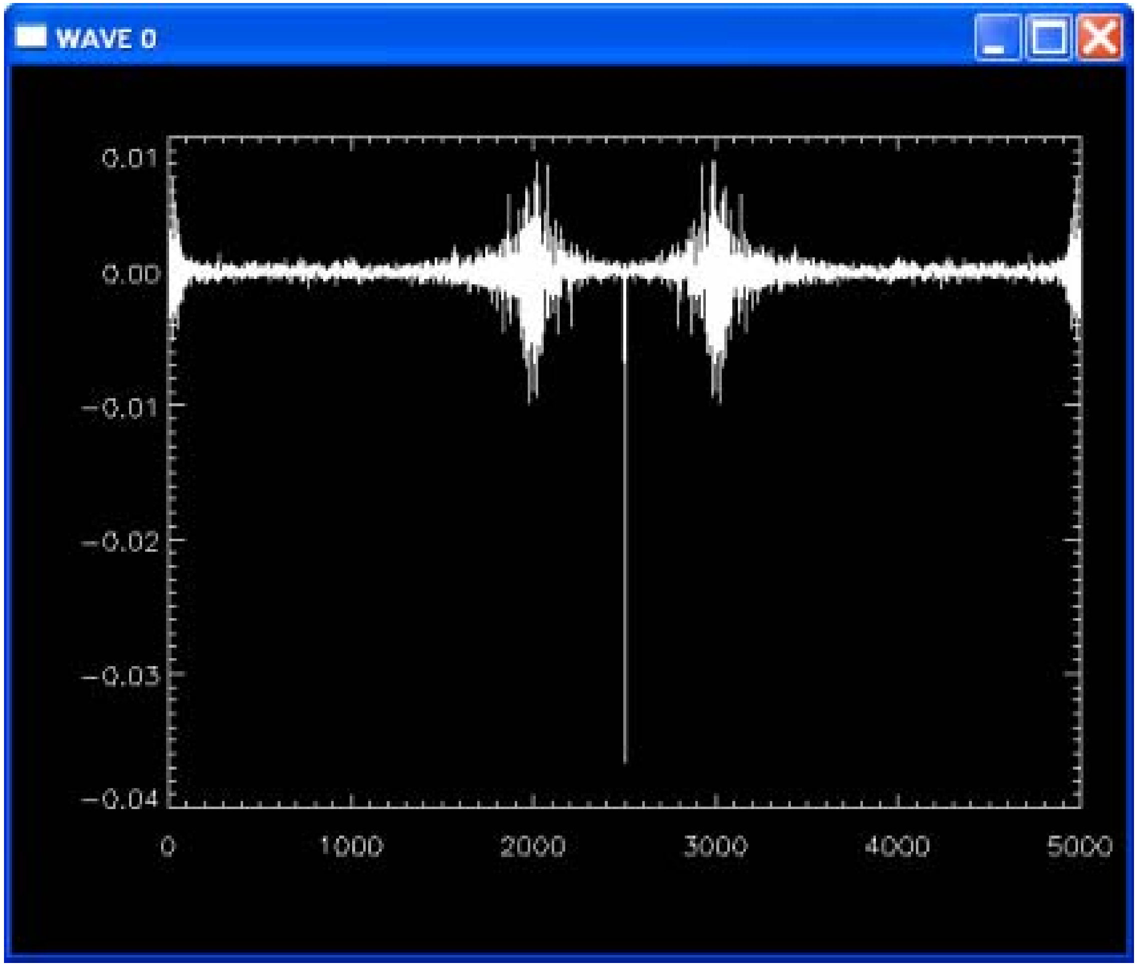
Many math packages do FFT's including PVwave. When you do an FFT of one of your signals what do you get back?

Suppose I go into the lab and digitize a signal of B(t) where B is one component of the magnetic field. B(t) is due to waves. B(t) has 5001 time steps. The data looks like:



The PVwave command to do the FFT is   > bfft=fft(B,-1)

When you do this what do you get?



This does not look like a spectrum at all. Also it looks like the result is mirrored about the point in the center. What is going on ?