

# LAPTAG

Lecture notes on Waves/Spectra  
Noise, Correlations and ....

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Our digital data is a function of time  $x(t)$  and can be represented as:

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \quad \omega = 2\pi f$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos n\omega t dt \quad n = 0, 1, 2, 3, \dots$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin n\omega t dt \quad n = 1, 2, 3, \dots$$

But since time is not continuous after we digitize  $t = m\Delta t$ ,  $\Delta t$  are the timesteps we digitize at:

$$x_m \equiv x(m\Delta t) = a_0 + \sum_{m=1}^{\frac{N}{2}} \left( a_n \frac{\cos n\omega m}{N} + b_n \frac{\sin n\omega m}{N} \right) \quad \omega = 2\pi f$$

Note the time has been replaced by  $m/N$  and the limit does not go to infinity but to  $N/2$  (the longest period we can measure due to the Nyquist limit!).

# Prescription

- We will use infinite series to represent all mathematical functions  
(note your data is a function  $x(t)$ , **you just happen to measure it!** )
- Instead of saying that the series are sums of sines and cosines we will represent them as series of  $A_n \sin(2\pi nt + \Phi_n)$
- Sometimes instead of using  $A_n \sin(2\pi nt + \Phi_n)$

we can use complex numbers and write each term as

$$x_n(t) = A_n e^{i(2\pi nt + \Phi_n)}$$

Note  $x_n(t)$  is the  $n^{\text{th}}$  sine term in the time series , not the  $n^{\text{th}}$  time step!!!!

You will see how we do this today!

The weights  $a_n$  and  $b_n$  are also discrete and the formulas:

$$a_n = \frac{2}{T} \int_0^T x(t) \cos n\omega t dt \quad n = 0, 1, 2, 3, \dots$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin n\omega t dt \quad n = 1, 2, 3, \dots$$

This requires  $n^2$   
operations!

Become:

$$a_n = \frac{2}{N} \sum_{m=1}^N x_m \cos \frac{n\omega m}{N}$$

$$b_n = \frac{2}{N} \sum_{m=1}^N x_m \sin \frac{n\omega m}{N}$$

$$a_0 = \frac{1}{N} \sum_{m=1}^N x_m = \langle x \rangle = 0$$

For  $a_0$  we assume that the signals are made of waves and the average value for all waves is zero ( $\langle x \rangle$  is the average value of  $x$ )

Since finding the spectra is very important mathematicians have found a way to rapidly perform these mathematical operations and get the same result. This is called the:

# *Fast Fourier Transform*

How this is implemented is complicated and you may learn it one day in an advanced math course in college but the bottom line is the FFT requires

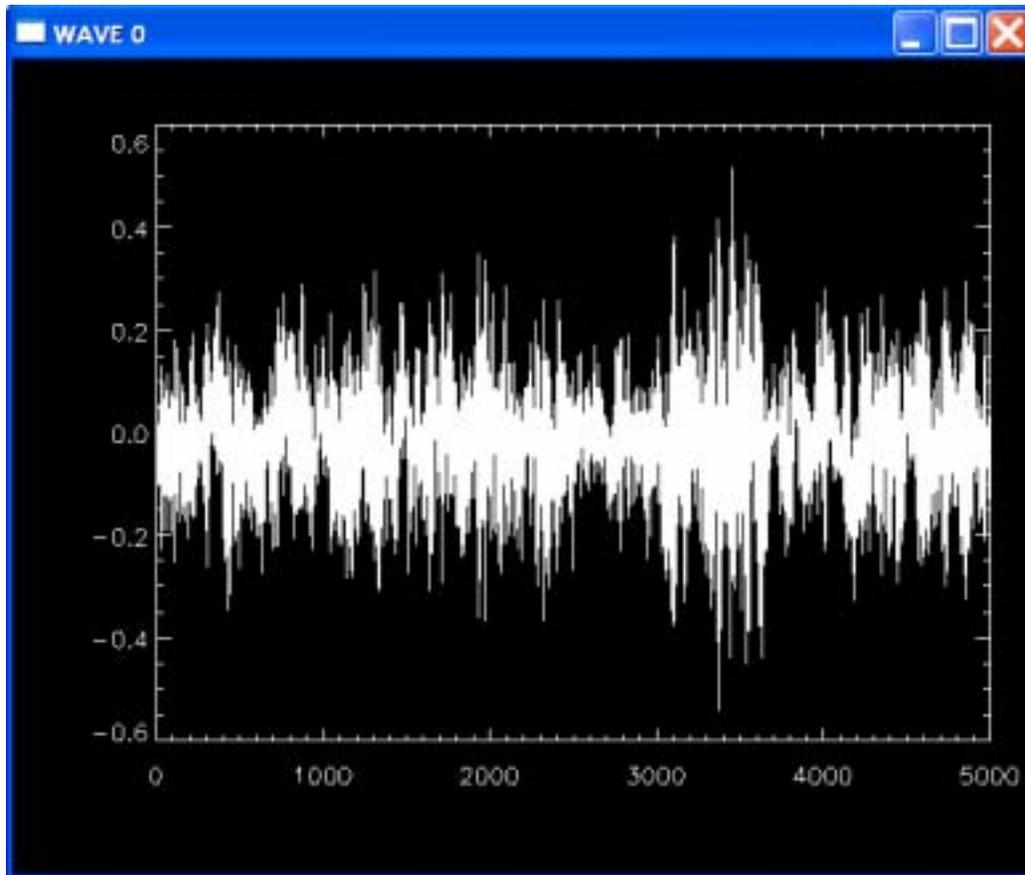
number of operations =  $N/2^p$  for data sets where the number of time steps is a power of 2 namely  $N=2^p$ .

Since  $2^{10} = 1024$  , number of operations =  $1024/(2^{*10})=52$

# **This is a big win!**

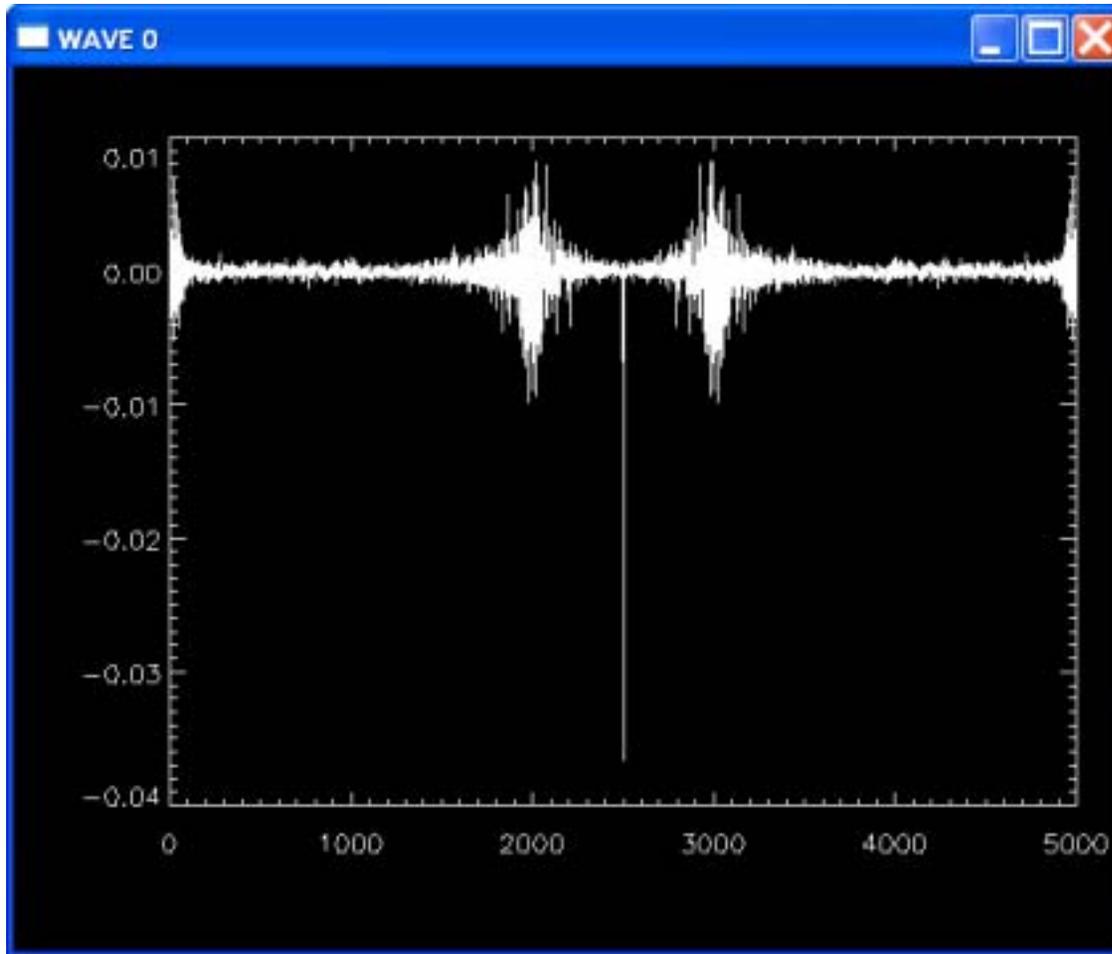
Many math packages do FFT's including PVwave. When you do an FFT of one of your signals what do you get back?

Suppose I go into the lab and digitize a signal of  $B(t)$  where  $B$  is one component of the magnetic field.  $B(t)$  is due to waves.  $B(t)$  has 5001 time steps. The data looks like:



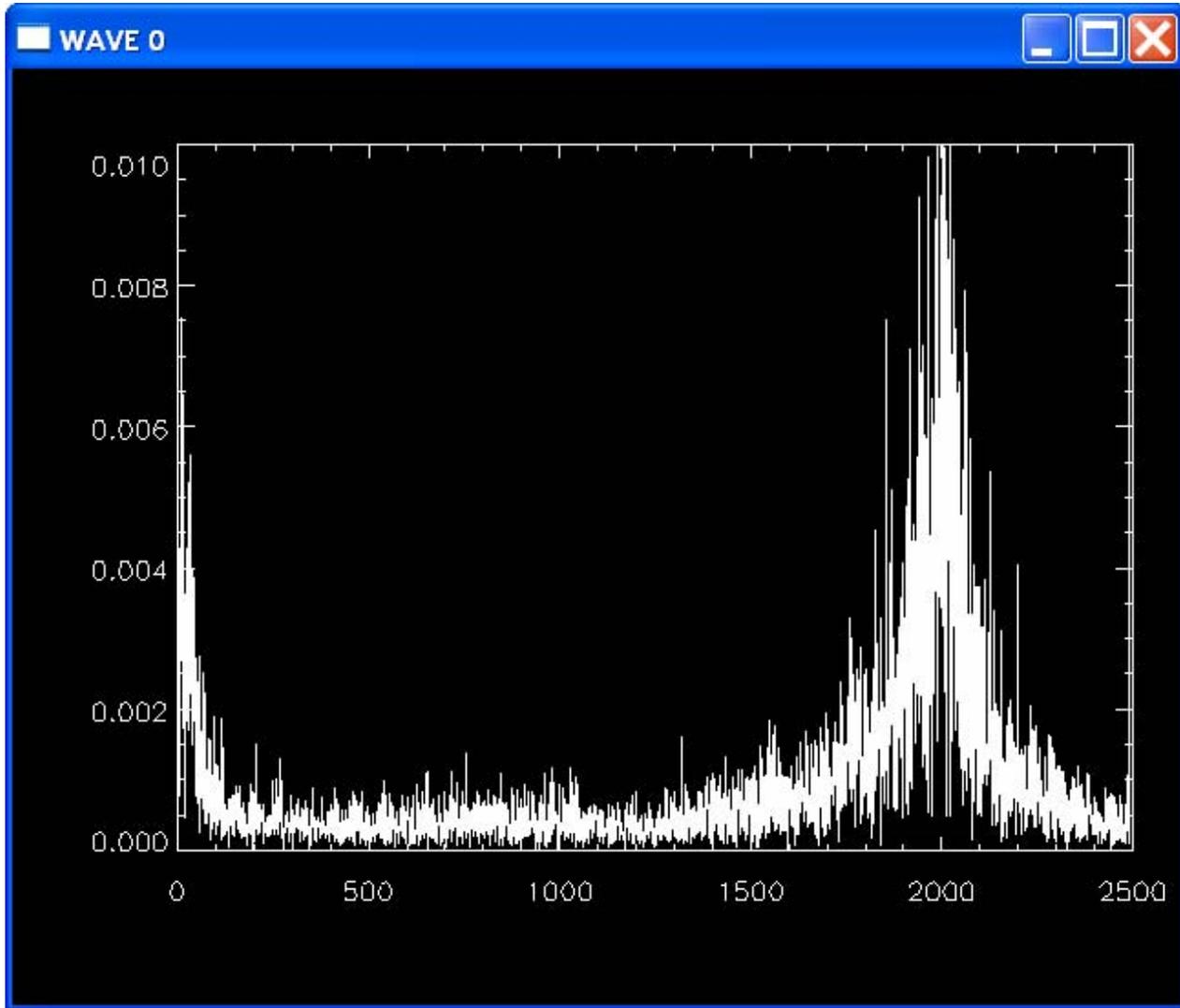
The PVwave command to do the FFT is `> bfft=fft(B,-1)`

When you do this what do you get?



This does not look like a spectrum at all. Also it looks like the result is mirrored about the point in the center. What is going on ?

The power spectrum which shows us the power of the wave as a function of frequency. There is a large peak at 2000!



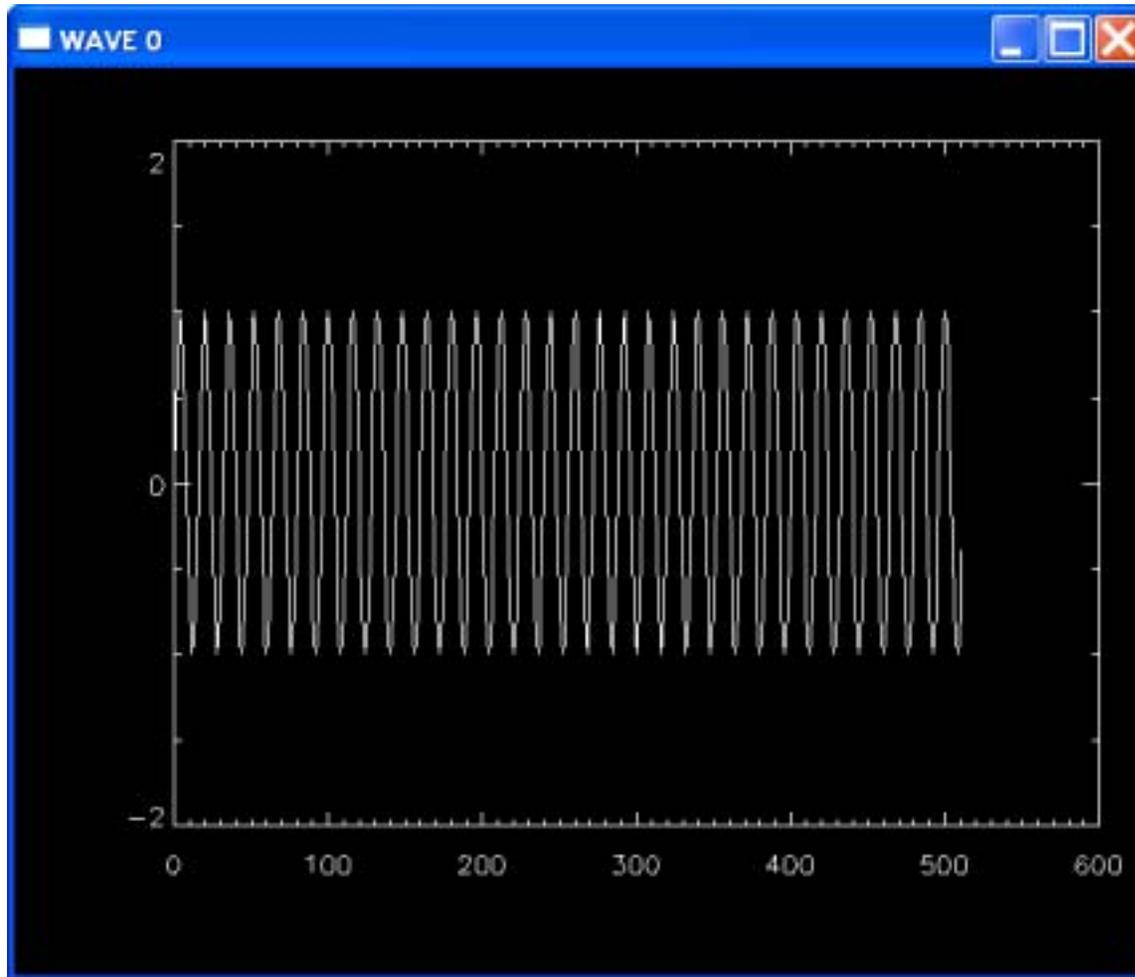
Lets find out how to get from the complex fft fo here, and what the axis mean

The only way to find out is find the FFT of a function that we know all about.

Lets do a sine calculated for 512 time steps.

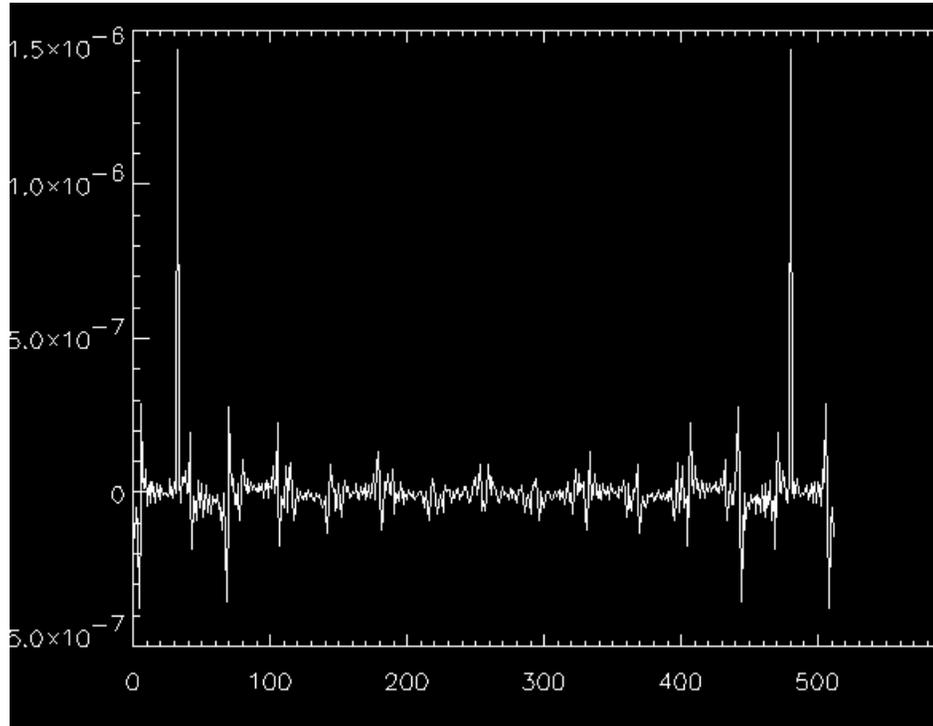
```
;
; A PVwave program to play with spectra
; a single sine wave to see what's what
; W. Gekelman Dec 2003
;
datin = fltarr(512) ; declare array (make space for it)
for i = 0 , 511 do begin ; 512 points
  datin(i) = 1.0*sin(2*!pi*float(i)/16.0)
endfor
plot, datin ; plot the sine wave
dffft = fft(datin,-1) ; take fourier transform
;
; examine fft (which is complex)
;
plot, dffft
; examine power in fft abs(dffft)=sqrt(re(dffft)^2+Im(dffft)^2)
plot, abs(dffft)
plot, abs(dffft(0:256)) ; look only in allowed Nyquist part
plot, abs(dffft(0:64)) ; look at the beginning
;
; examine what the real and complex parts look like
;
plot , Real(dffft)
plot, Imaginary(dffft)
end
```

What does the sine wave look like over the 512 timesteps ?



It has 16 cycles... *WHY?* Check the equation.

## What does the Fast Fourier Transform look like?

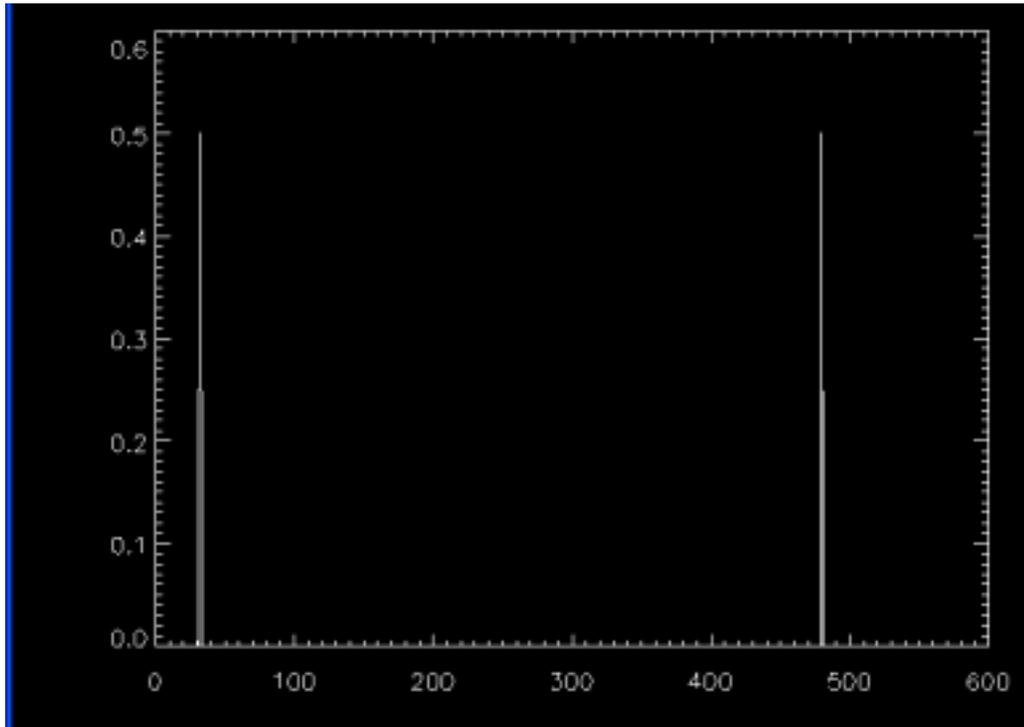


This looks strange because the FFT returns both the real and imaginary part of the function. If you ask `wave` it will tell you that `dfft` is a complex array of 512 points. It is symmetric about the midpoint ( $x=256$ ). What do we do next?

Lets calculate the power spectra (we will discuss what this is later) by:

$$power - spec = \sqrt{\text{Real}(fft)^2 + \text{Imaginary}(fft)^2}$$

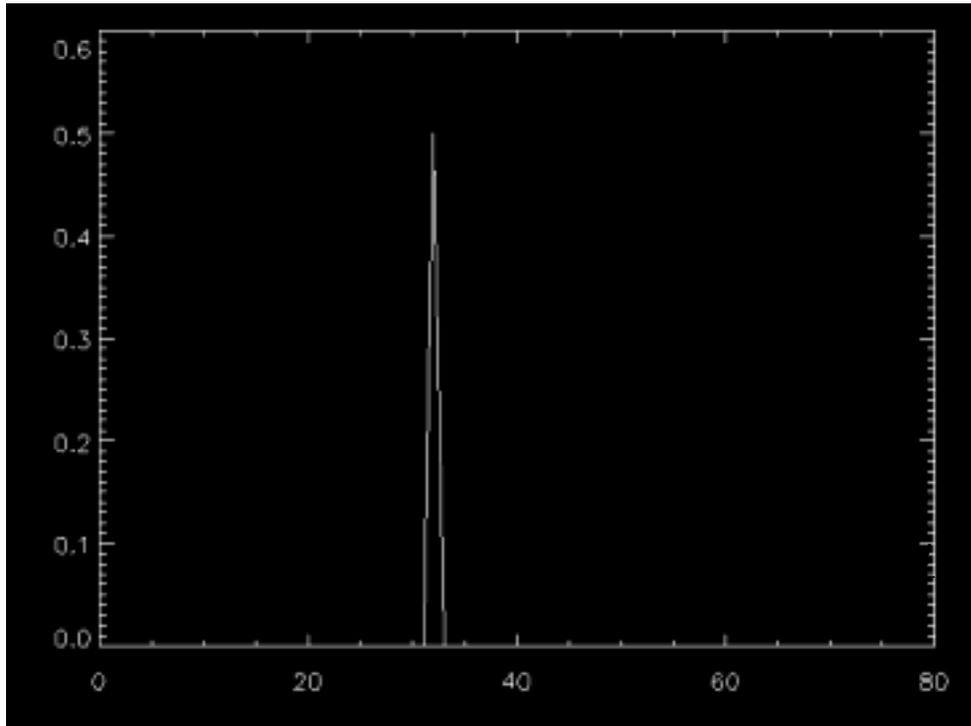
Remember if  $y=a+ib$  then  $y^* y = \sqrt{a^2 + b^2}$



This is the power spectra. Note there are two peaks still symmetric about  $x=256$ .

Each has a value of  $\frac{1}{2}$  since the Fourier transform assigns half the power to negative frequencies and half to positive frequencies.

Lets look at the low frequency mode.



The spectrum has a peak at  $t = 32$ , Does this make sense?

$$y = \sin\left(\frac{2\pi t}{T}\right) = \sin\left(\frac{2\pi t}{16}\right)$$

$t = 1, 2, 3, \dots$  (discrete steps)  
 ; when  $t = 16$  we go through 1 cycle

Suppose 512 steps is 1 second

1 time step is  $1/512$  seconds

If we have 16 cycles in 512 steps then each cycle is .01325 sec,  $f = 32$  Hz !!!

What's the lowest frequency we can see?

Answer flow =  $1/(2dt)$  (Nyquist limit...last lecture)

1/2 cycle in 512 steps.  $f_{\text{lowest}}$  is 2 second.

What's the highest frequency we can see?

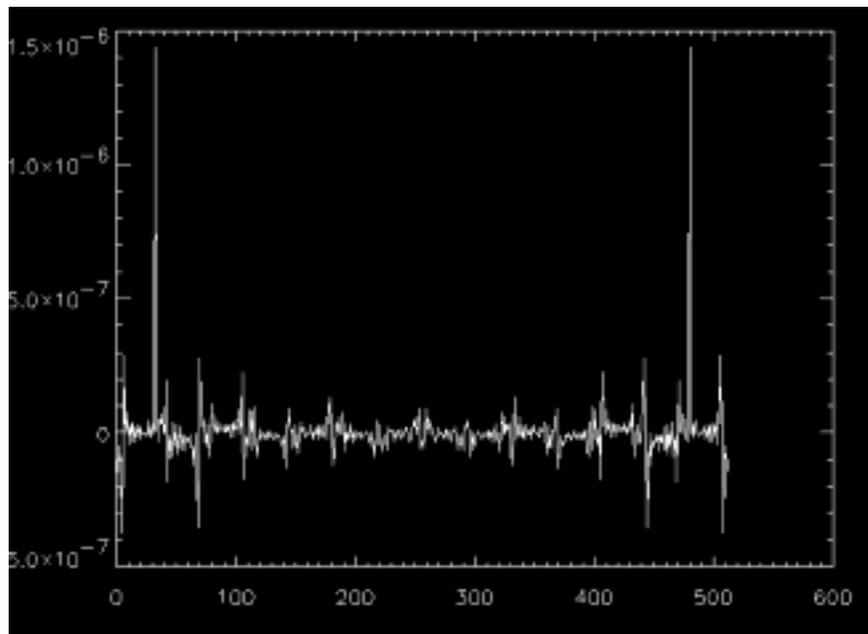
It takes at least 2 steps to define 1 cycle

2 time steps =  $2/512$  seconds = .00390635 seconds

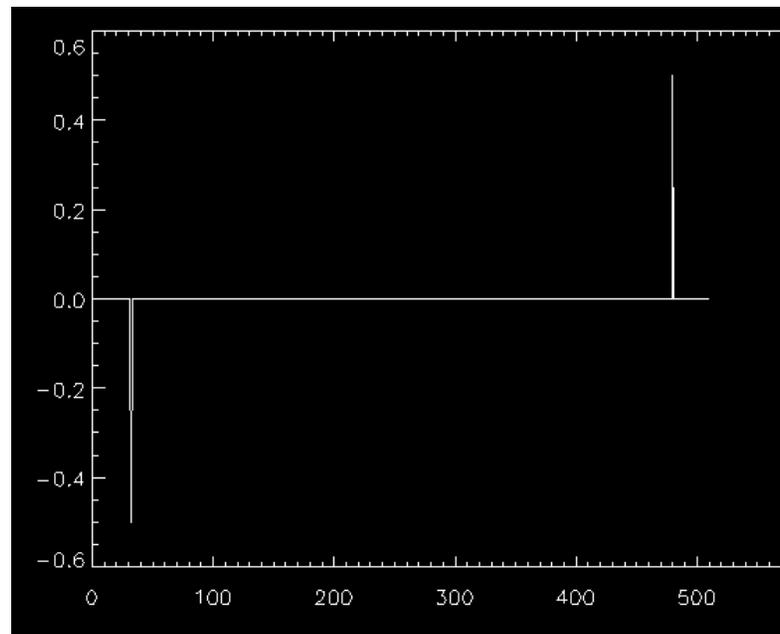
$f_{\text{(highest)}} = 256$  Hz

Note that it is no accident that the spectrum goes from 0-256 instead of 0-512. This is because the highest frequency is determined by the requirement that it takes two points per cycle at the very least to determine a wave period.

## The Real and Imaginary Parts of the Fourier Transform:



Real Part

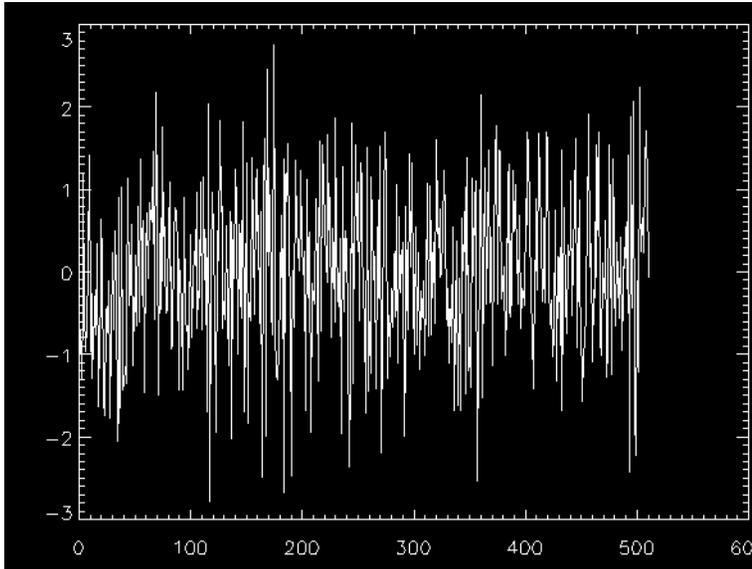


Imaginary Part

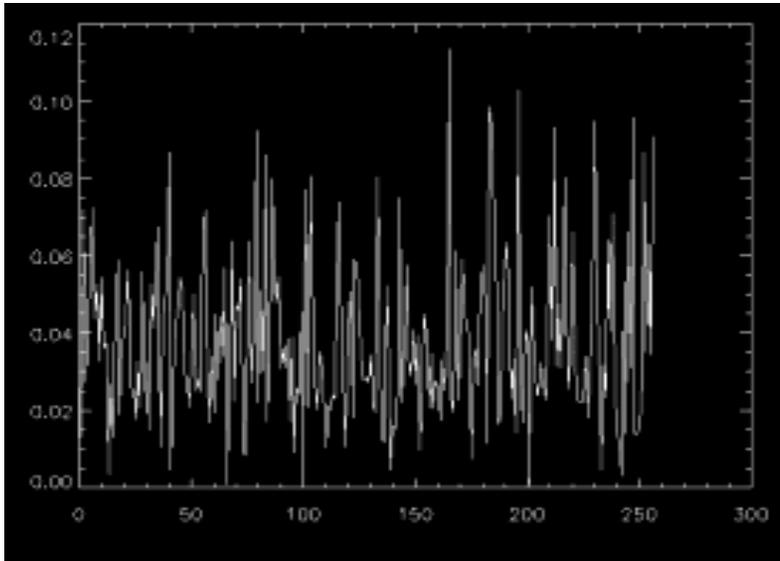
Since  $y = A\cos\theta + iB\sin\theta$  ( $a + ib$ )

In this case  $y = \sin\theta$  (not some other complicated thing) so that the imaginary part happens to be our function and the Real part is a million times smaller. Its not exactly zero because the data is digital not continuous in time.

What is the data is truly random? i.e. pure noise

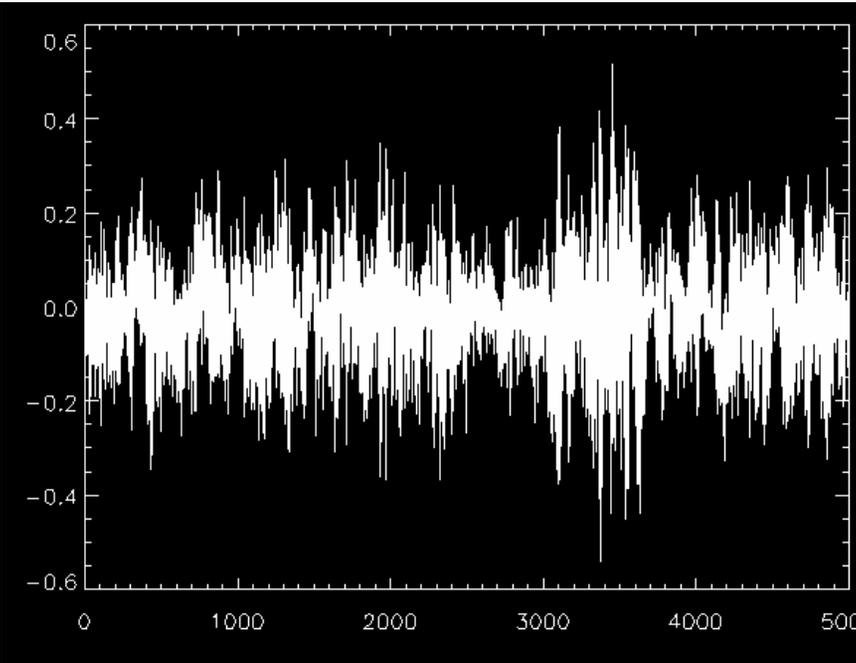


Random data generated by a random number generator



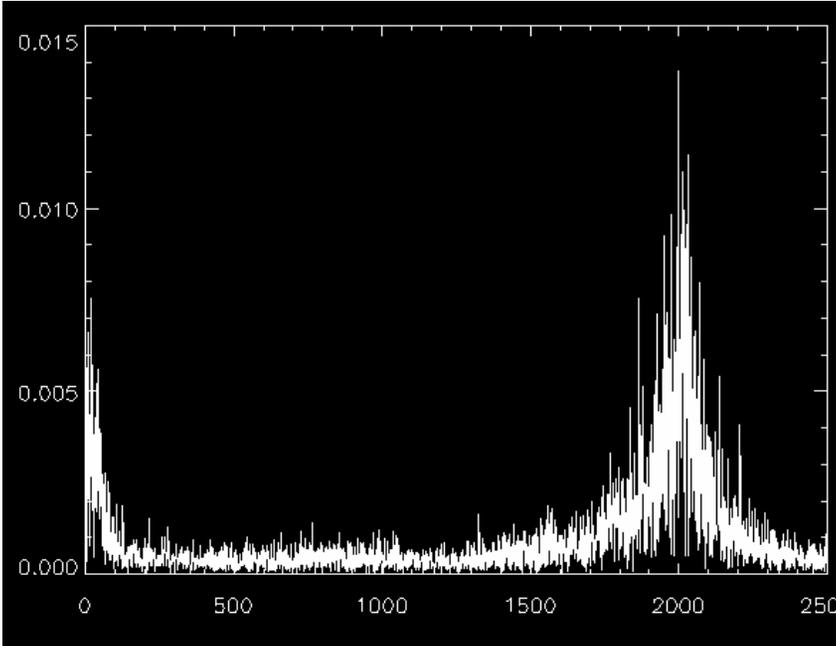
The power spectra of the completely random data. All frequencies are present and they are, in turn, randomly distributed

# Once again lets look at the magnetic field data and the Fast Fourier Xform



The magnetic field data looks like noise just as the random data in the previous slide.

The magnetic field spectrum has peaks and looks quite different than the previous picture. There are magnetic waves in this data at frequency 500 and 2500 (in the units of this graph)



If we have a complete spectrum (the real and imaginary parts) the **INVERSE** Fourier Transform allows us to go from frequency space into temporal space.

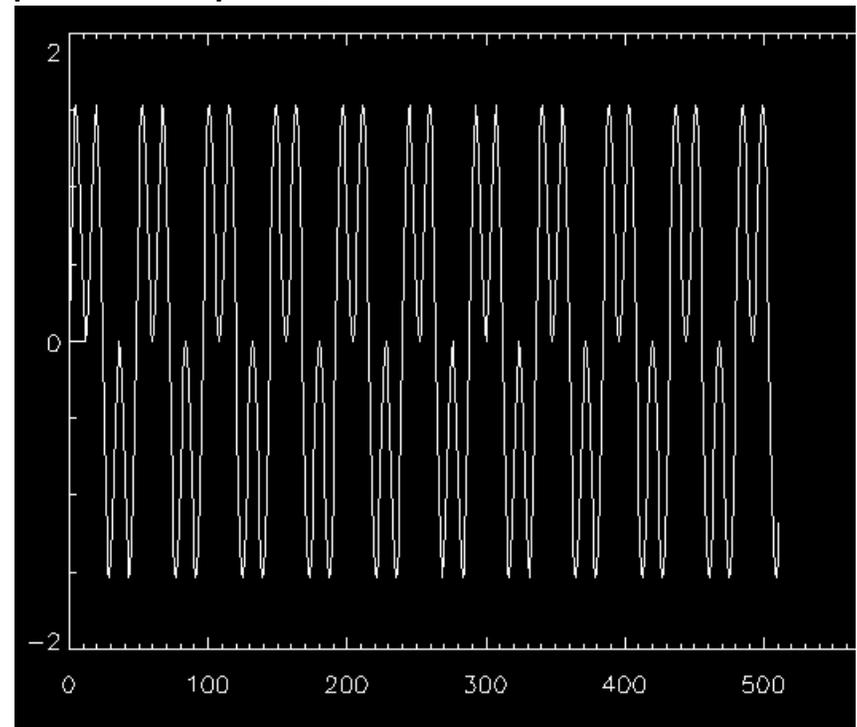
Symbolically 
$$\tilde{x}(t) = FFT^{-1}[\tilde{x}(f)]$$

Where  $FFT^{-1}$  is the inverse FFT and both  $x(t)$  and  $x(f)$  are complex, they have the amplitude and phase stored.

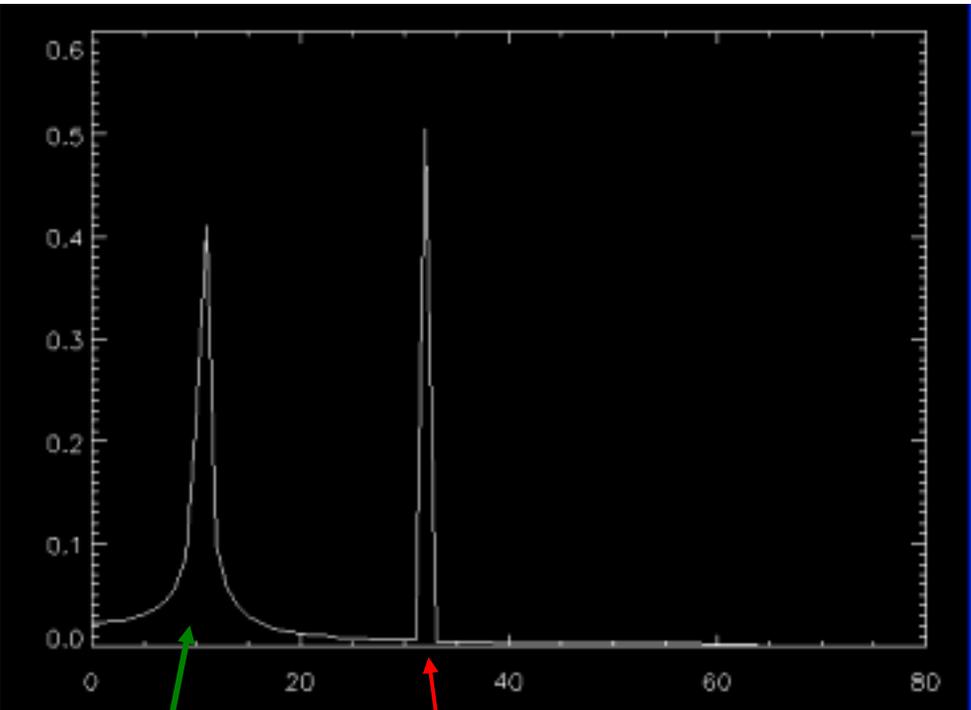
What good is it? Suppose we have found our spectra in the previous slide and we want to get rid of all of the signal except for the peak, and see what that looks like in time.

This is two sine waves added together

$$y = \sin(2\pi t/16) + \sin(2\pi t/48)$$



The spectrum is:



Low frequency

High frequency

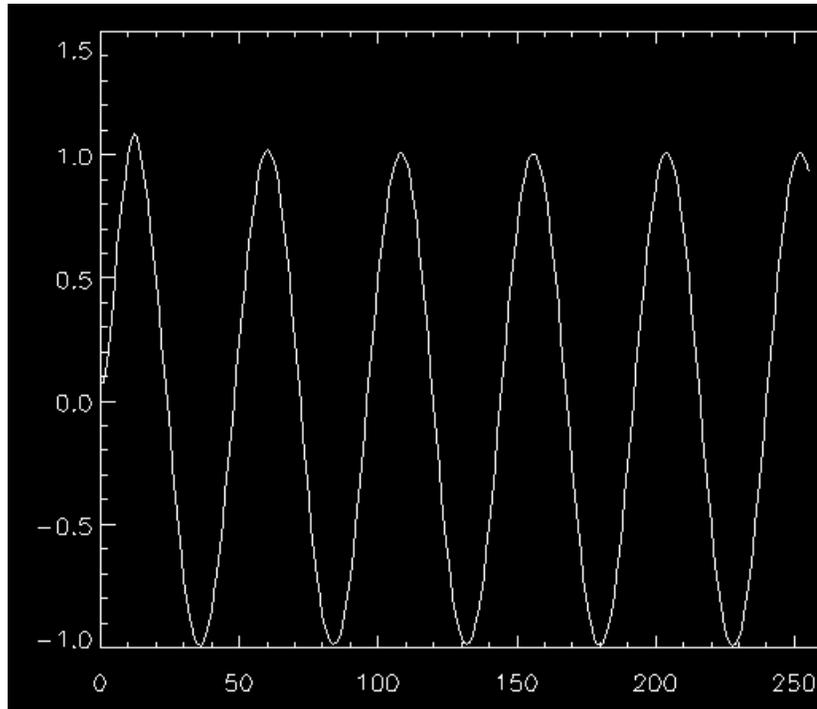
If we go into the fft of the original signal and cut out the high frequency Component (we have to do this with the complex fft) and then do an inverse FFT

```
In wave > bnew = fft(bfft,+1)
```

+1 means do inverse fft

New array (function of time)

Complex fft with high frequency cut out



After digital filter. We got rid of the high frequency sine wave. This is the real part. Note the imaginary part of  $b_{new}$  is close to zero at all time for this one!

The spectra can tell us if there is a signal present in the data and what the frequencies of the waves which are in the noise. Are there other techniques we can use that tell us more about the waves in the noise?

1) How long do they last before they disappear?

**Answer** Autocorrelation functions

2) Can we separate one wave from the noise and look at it?

**Answer** Digital Filtering and Cross Spectral Functions

3) Can we find the dispersion relation (how the frequencies and wavelengths are related) from studying the noise?

**Answer** Two point Correlation Functions.

Let us examine these techniques one by one