

Fourier analysis:

Fourier proved that any “well behaved” mathematical function can be written as an infinite series of sines and cosines. Actually this holds true for all orthogonal functions but we wont discuss that now.

If $f(z)$ is a function then:

$$(1) F(z) = B_0 + \sum_{m=1}^{\infty} A_m \sin(mkz) + \sum_{m=1}^{\infty} B_m \cos(mkz)$$

Fourier found a method to find all the A's and B's in series. Without proving it (you will see this later in a math class) .

$$B_0 = \frac{1}{\lambda} \int_z^{z+\lambda} F(z) dz$$

$$A_m = \frac{2}{\lambda} \int_z^{z+\lambda} F(z) \sin(mkz) dz$$

$$B_m = \frac{2}{\lambda} \int_z^{z+\lambda} F(z) \cos(mkz) dz$$

The integrals can be treated as infinite sums. For the A coefficients:

$$A_m = \frac{2}{\lambda} [F(z_0) \sin(mkz_0) + F(z_1) \sin(mkz_1) + F(z_2) \sin(mkz_2) + \dots + F(z_n) \sin(mkz_n)]$$

where $z_1 - z_0 = \epsilon$ and $\epsilon \rightarrow 0$

What about a Fourier series for something that depends on time only. In that case we can simply replace $mkz \rightarrow m\omega t$. The mathematics does not care.

As we mentioned before we can write the solution to the wave equation as the real part of $y = (Ae^{i(kx - \omega t)})$. This is actually one term in the general Fourier

series:

$y = \sum \left(A e^{i(nkx - n\omega t + \phi_n)} \right)$ The nk 's for example are harmonics of the wavenumbers and as long as $\frac{\omega}{k} = v$ each term satisfies the wave equation. There can be a different phase term for each n .

What do all these symbols mean? Z is a spatial position, k is a constant (it turns out is the wavenumber $\left(k = \frac{2\pi}{\lambda} \right)$ and m is an integer. The A_m and B_m are the Fourier coefficients. They tell you how much weight (amplitude) goes along with each (m^{th}) term.

As an example the following is the Fourier expansion of a square wave:

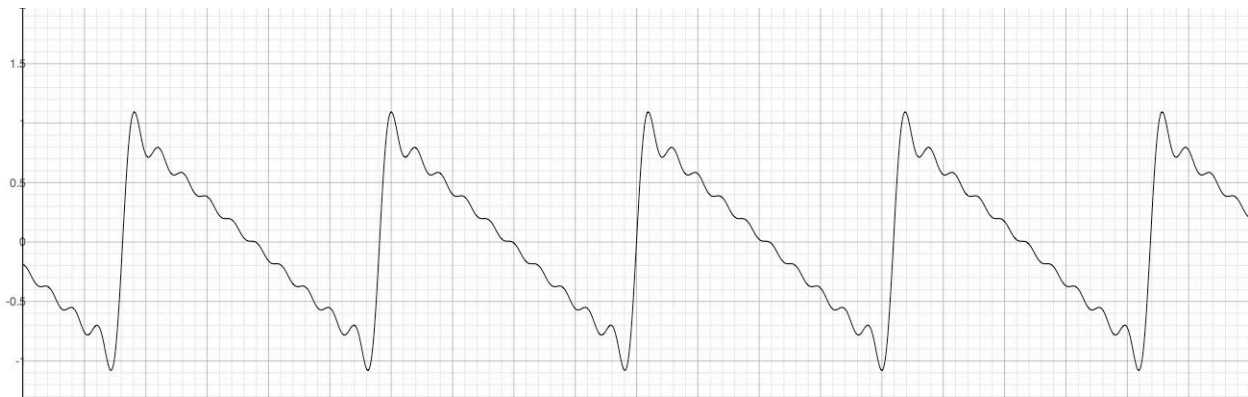
Here we have included the first 10 k 's to build the square wave. Note that there are no cosine terms (all the integrals for the B 's are zero). Also the function is periodic as all sines and cosines are. This makes sense as the waveform is zero at $x = 0$ just as the sine waves are. If one was interested in only the first pulse then ignore all the others.

In another example consider a sawtooth wave. The expression is now

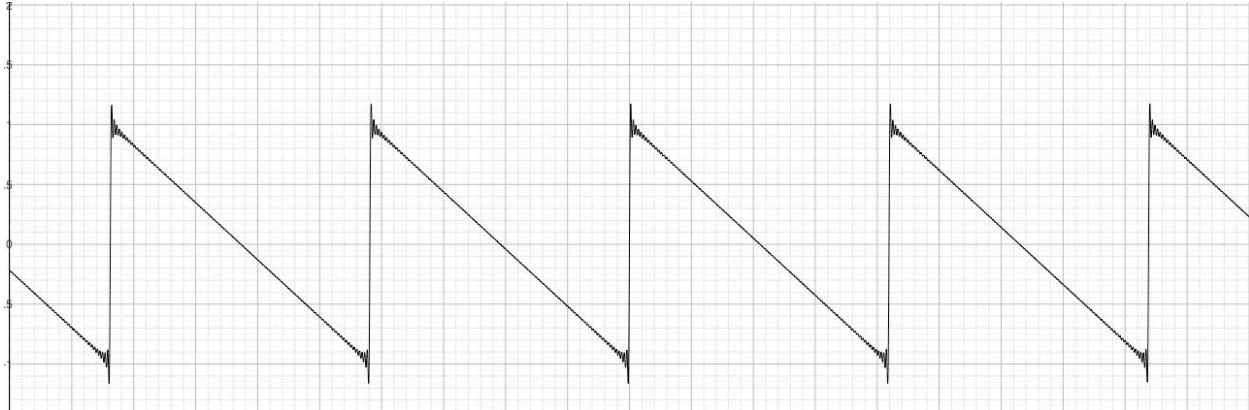
$$y = 0.63\sin(kx) - .031\sin(2kx) + 0.21\sin(3kx) - .015\sin(4kx) + 0.12\sin(5kx) - .01\sin(6kx) + \dots$$

This works out to be $y = \frac{2}{\pi} \sum_{\alpha=1}^{\infty} \frac{\sin(\alpha kx)}{\alpha}$

As we add more terms the approximation becomes better and better!



N=10 terms



N = 100 terms

How much work is Fourier analysis. Suppose you had to write a computer program to do it? We collect a data set that has N time samples. Now we want to find the Fourier coefficients. T is the period that we take data (100 usec for example) and ω is the angular frequency

$$A_n = \frac{2}{T} \int_z^{z+T} X(t) \sin(n\omega t) dt \quad n = 0,1,2,3,4,\dots$$

$$B_n = \frac{2}{T} \int_z^{z+T} X(t) \cos(n\omega t) dt \quad n = 1,2,3,4,\dots$$

But A and B are discrete because the function is not continuous but is made out of digital data. Therefore the integrals become

$$A_n = \frac{2}{N} \sum_{m=1}^N x_m \sin\left(\frac{n\omega m}{N}\right)$$

$$B_n = \frac{2}{N} \sum_{m=1}^N x_m \cos\left(\frac{n\omega m}{N}\right)$$

$$B_0 = \frac{1}{N} \sum_{m=1}^N x_m = \langle x \rangle$$

(1) For every n and m evaluate $\theta = \frac{2\pi nm}{N}$

(2) find $\sin\theta, \cos\theta$

(3) compute $x_m \sin\theta, x_m \cos\theta$

(4) sum all the terms $m=1,2,3,4 \dots N$

(5) go to the next value of n and start again

This takes approximately N^2 multiply and add operations. If $N=1024$ then 1,048,576 operations are required! The Fast Fourier Transform was discovered by

Mathematicians at Bell Labs. It radically cuts down on the number of operations without losing accuracy. The number of operations goes to $N/2^p$ if the data set has $N=2^p$ data points. (Modern versions don't restrict the FFT to powers of 2.) If $N=1024$ then $p=10$ and the number of operations is $Num = \frac{1024}{(20)} = 52!$ IDL has a subroutine that

does fast fourier transforms. Suppose the data set of interest is X

`X =fltarr(1024)` ; has 1024 numbers

; read in data set

`y=fft(X,-1)` ; this does the Fourier transform

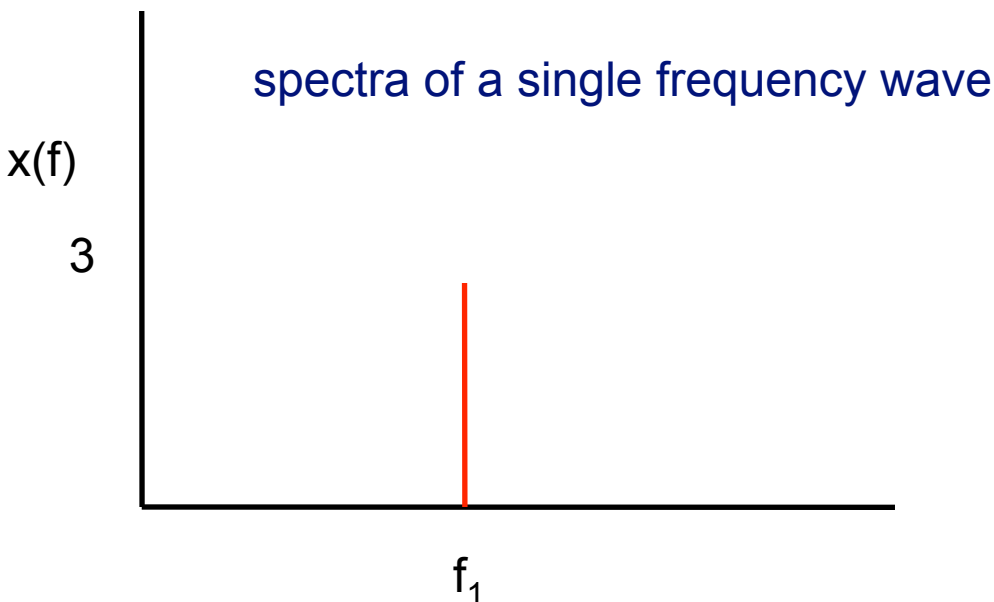
If ask what Y is

help, Y

you will get

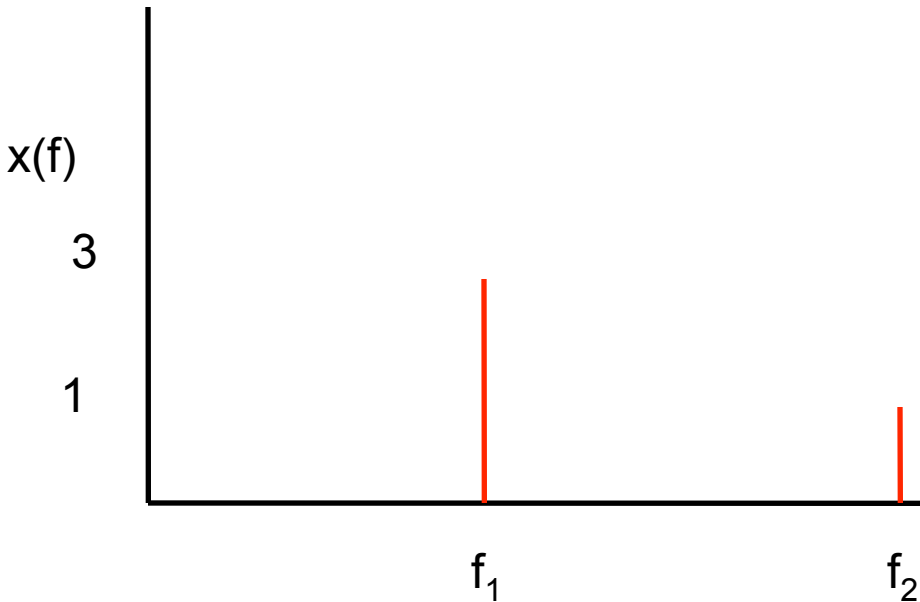
> `Y=complex(512)`. ; The FFT returns half the number of complex numbers. Why?

The Fourier decomposition allows us to introduce the notion of a spectrum. Let us look at a function in the time domain such as a trace on your oscilloscope. Suppose the signal is composed of a single sine wave: $x(t) = 3\sin(2\pi f_1 t)$. Instead of plotting we sine wave suppose we plot every sine wave that the function is made out of. In this simple case the plot would look like:



We have given up the temporal information to make this plot and we see it in frequency space. In the case of 2 sine waves $x(t) = 3\sin(2\pi f_1 t) + \sin(2\pi f_2 t)$ the spectrum would be

spectra of two frequency wave



If there are many frequencies in the waveform the spectrum becomes continuous:

The next step is to do the mathematics of Fourier analysis using complex numbers. In the study of spectra and their close cousins, correlation functions we frequently go back and forth between complex numbers and sines and cosines. To do this we use Euler's (DeMoivre's) theorem:

$$(1) e^{i\theta} = \cos\theta + i\sin\theta \quad \text{where } i = \sqrt{-1}.$$

Also note that since any complex number may be written as:

$$\tilde{z} = a + ib \quad \text{The } \sim \text{ means complex}$$

$$\text{also: } \tilde{z} = ze^{i\theta}$$

$$z = \sqrt{a^2 + b^2} \quad \text{magnitude of } \tilde{z}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) \quad \text{phase}$$

Using equation (2) we easily show that if $\theta = \omega t$

$$(2) \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Therefore we can write the n th term of in a Fourier series $a_n \cos(\omega_n t + \Phi_n)$ as

$$\text{Re}(a_n e^{i(\omega_n t + \Phi_n)}) = \text{Re}(a_n e^{i\Phi_n} e^{i\omega_n t}) = \text{Re}(\tilde{a}_n e^{i\omega_n t}) \quad \text{where } \tilde{a}_n \text{ is a complex number with the}$$

phase built in. The solution to the wave equation depends on space and time

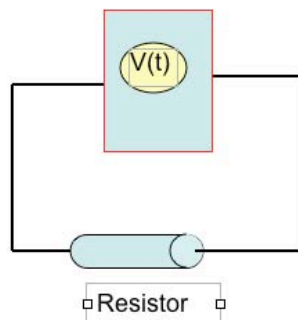
and by extension it can be written as $\Psi = \tilde{a}e^{i(kx-\omega t)}$ for a single frequency wave traveling in the x direction. A superposition of waves would have many w's and k's.

$$\langle X \rangle = \frac{1}{N} \sum_{n=1}^N X_n \quad \text{mean}$$

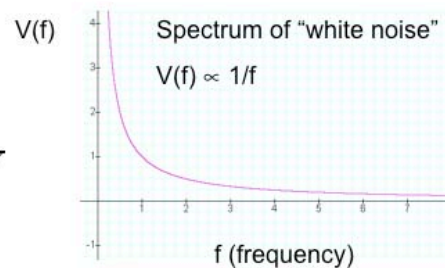
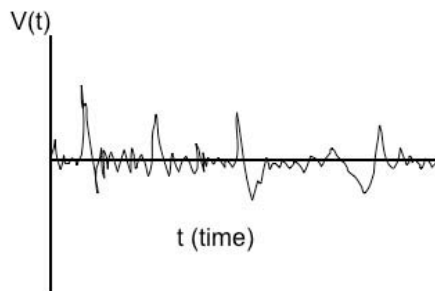
$$\sigma = \sqrt{\frac{1}{N-1} \sum_{n=1}^N (X_n - \langle X \rangle)^2} \quad \text{standard deviation}$$

Random data,

Some noise is truly random and is not made up of waves



Take a voltage and measure the voltage as a function of time across a resistor, V(t). Its average is zero because there is no battery in the circuit but:



Autocorrelation functions etc...

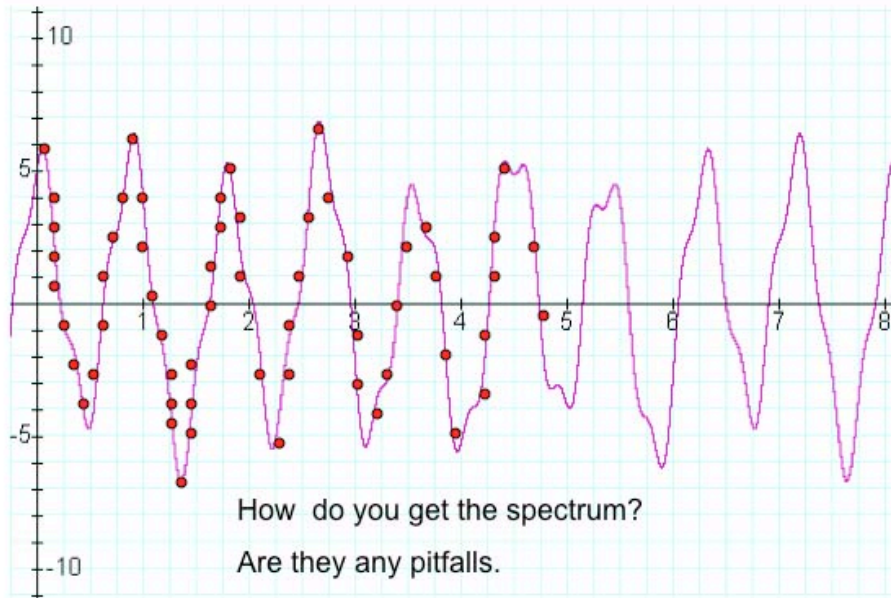
$$skew = \frac{1}{N} \sum_{n=1}^N \left[\frac{X_n - \langle X \rangle}{\sigma} \right]^3$$

$$kurtosis = \frac{1}{N} \sum_{n=1}^N \left[\frac{X_n - \langle X \rangle}{\sigma} \right]^4 - 3$$

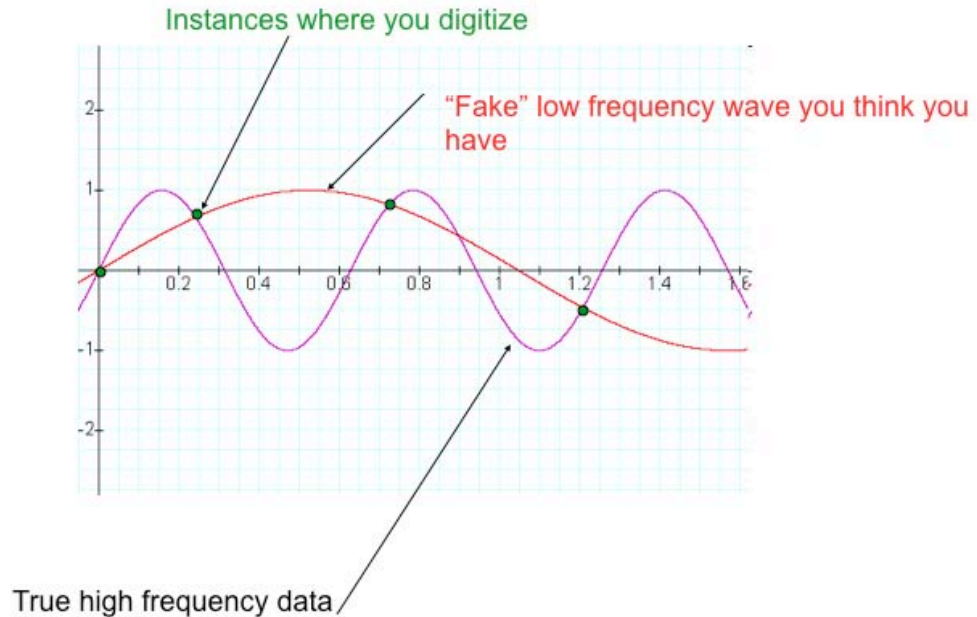
$$G(k) = \frac{\sum_{n=1}^{N-k} (X_n - \langle X \rangle)(X_{n+k} - \langle X \rangle)}{\sum_{n=1}^{N-k} (X_n - \langle X \rangle)^2}$$

Autocorrelation

The purple continuous signal is the actual waveform but you digitize it at discrete times where the red dots are.



The biggest pitfall is called **aliasing**. Suppose you have a signal varying very rapidly in time and your digitizers have a time response which is too slow:



$$L_k = \frac{1}{2(N-k-m+1)} \sum_{n=m}^{N-k} \ln \sum_{j=0}^{m-1} (X_{n-j+k} - X_{n-j+k})^2 \quad \text{for Lyapunov Exponent}$$

Suppose we have a time series and we wish to find the nearest point X_i to another point X_n . This takes place in an m -dimensional time series

embedding. That is to say suppose there are N points in the time series for example N=10000. We then break the time series up into k series. If k = 10 then each series has 1000 points. L_k is the average logarithmic rate of separation of two points for the next k time steps. Note one could also take the data 10 different times each with 10000 points. The Lyapunov exponent is defined as

$\lambda = \frac{dL_k}{dk}$ Lyapunov exponent. We will introduce and use these quantities as the course proceeds.

If we look a spectra and see a signal present in the data in the form of spectral lines how can we learn more about the composition of the noise.
 How long does a signal last until it vanishes: Autocorrelation function.
 Can we separate one wave from the noise at look at it? Digital filtering and Cross Spectral Functions. Can we find a dispersion relation (how are the frequencies and wavelengths related?). : Two point correlation functions.

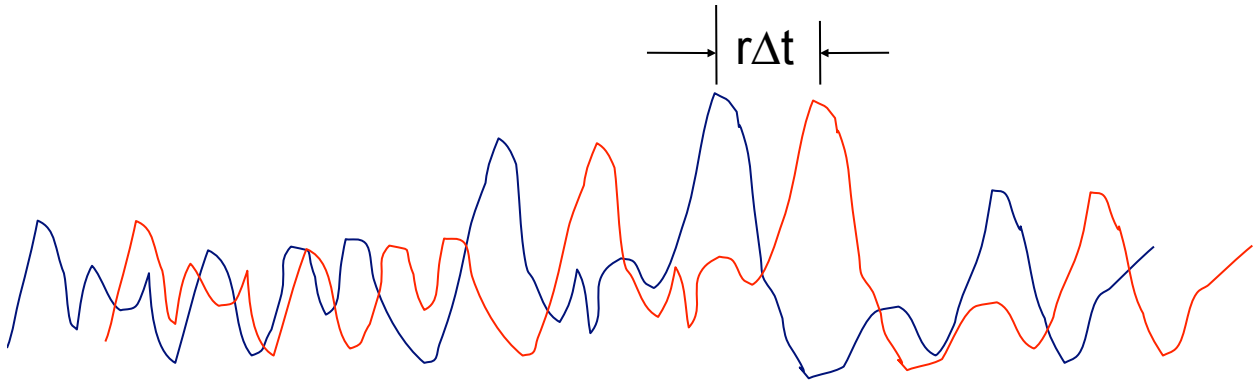
The autocorrelation function tells us the time over which a correlation in the noise exists. If there is a wave in the noise which is constantly being generated and moves without decaying the autocorrelation at the frequency of that wave will be very large.

$$(4) R_{xx}(\tau) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N x(t)x(t+\tau)$$

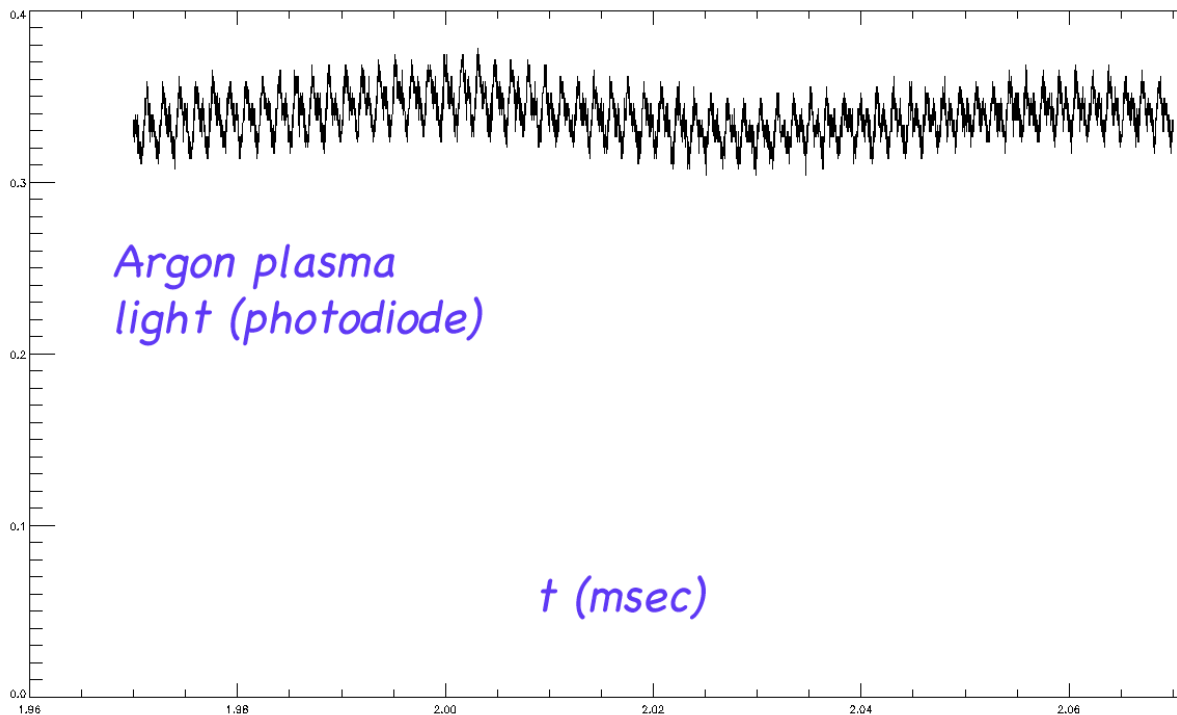
This is the case where x is a continuous function. Suppose it is digital data not continuous in t but at intervals $r\Delta t$

$$(5) R_{xx}(r\Delta t) = \lim_{N \rightarrow \infty} \frac{1}{N-r} \sum_{n=1}^{N-r} x(t)x(t+r\Delta t) \equiv \frac{1}{N-r} \sum_{n=1}^{N-r} x_n x_{n+1}$$

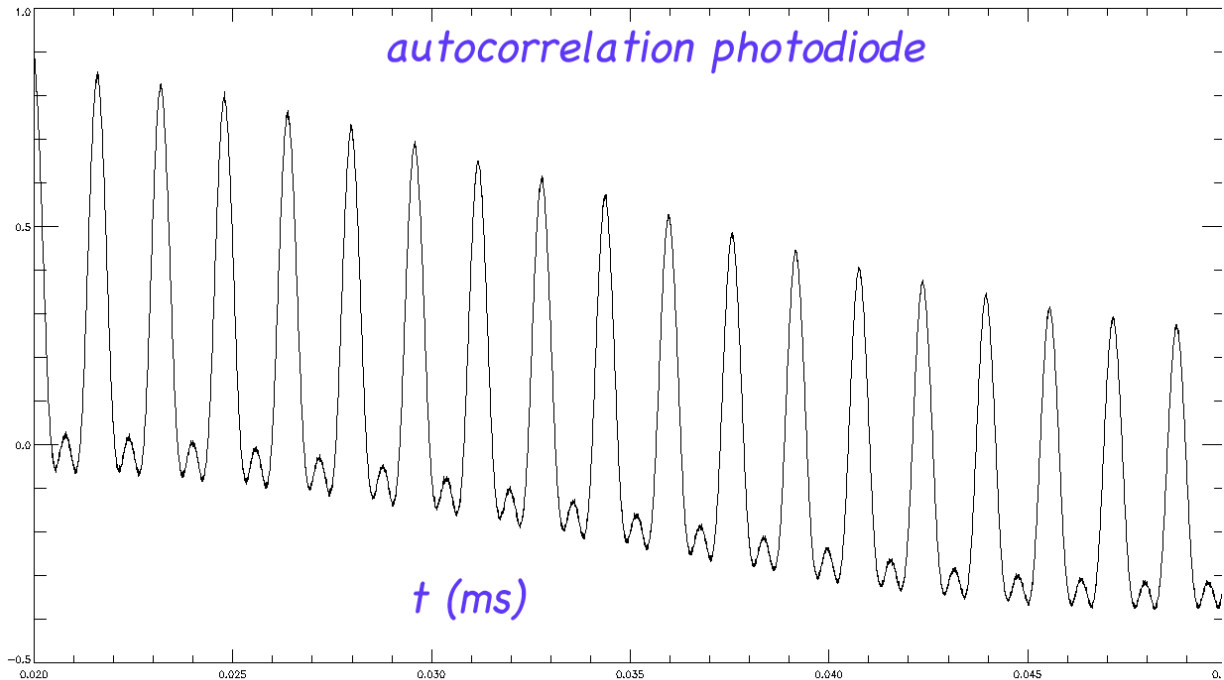
where $r = 0,1,2,3...m$ $m < n$
 What does this mean? Take two traces from separate probes.



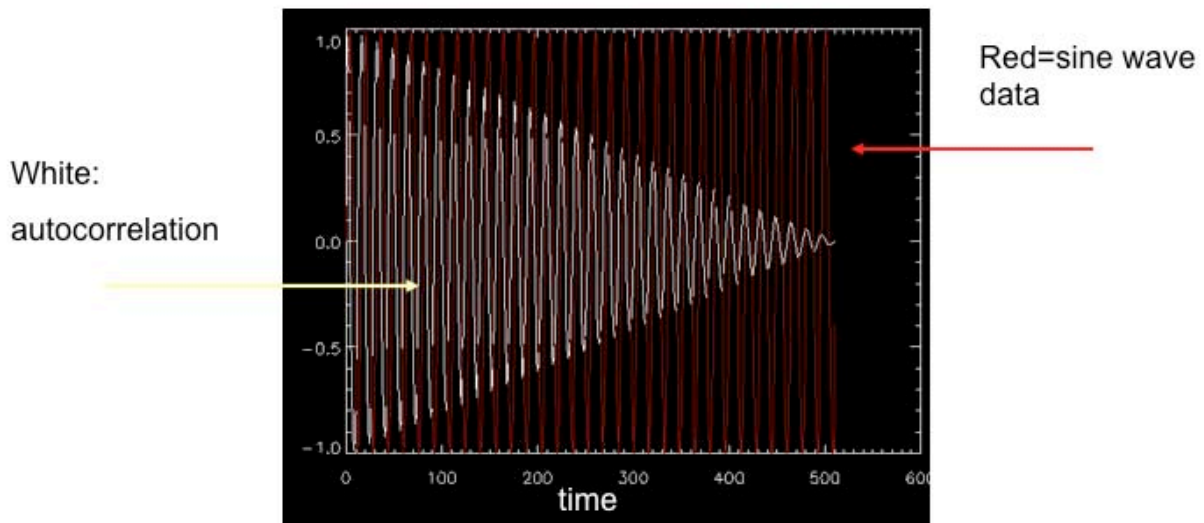
For any rDt multiply the two functions. Then slide the “red” data over by a timestep of Dt and multiply all the terms again. If there is only one probe then this becomes the autocorrelation data. Note that if $r=0$ then the guarantees the first term is 1. Consider light collected from the photodiode as a function of time shown in the figure below:



The autocorrelation function for the light is shown in the next figure. At zero lag time it starts as one. It's average slowly decays over 10s of microseconds but oscillates in time at the frequency of the RF in the inductive coil that makes the plasma. Why does the autocorrelation go away in time. Suppose we have a signal that is a perfect sine wave. If you now use the sine wave in equation 5 or 6 you will get a sine wave back. Try it. Suppose on the other hand you have digital data. The record does not go on forever. When you get to the end of it



you will now move your lag window to a place where data does not exist. The autocorrelation of necessity must die down as shown in the figure below.



To get around this one must take the longest possible record and don't let the lag window slide over the end of the data. One must be careful not to confuse a decaying signal which could reflect the physics of a process with decay due to a finite length record.

There is a mathematical relationship between the autocorrelation and the spectra.

$$(7) S_{.xx}(f) = \int_{-\infty}^{\infty} R_{.xx}(\tau) e^{-2\pi j f \tau} d\tau$$

Here R is the autocorrelation function which depends on time and S the frequency spectrum. This was proven by Von Neuman who is the father of Cybernetics.