Collisionless coupling of a high-β expansion to an ambient, magnetized plasma. I. Rayleigh model and scaling

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(Received 1 May 2017; accepted 22 August 2017; published online 12 April 2018)

The dynamics of a magnetized, expanding plasma with a high ratio of kinetic energy density to ambient magnetic field energy density, or \( \beta \), are examined by adapting a model of gaseous bubbles expanding in liquids as developed by Lord Rayleigh. New features include scale magnitudes and evolution of the electric fields in the system. The collisionless coupling between the expanding and ambient plasma due to these fields is described as well as the relevant scaling relations. Several different responses of the ambient plasma to the expansion are identified in this model, and for most laboratory experiments, ambient ions should be pulled inward, against the expansion due to the dominance of the electrostatic field. Published by AIP Publishing.

https://doi.org/10.1063/1.5029301

INTRODUCTION

When a plasma expands into an ambient magnetic field, part of the field is removed. In the case of kinetically driven expansions where the initial ratio of the kinetic energy density to the magnetic field energy density, or \( \beta \), is much greater than one, the field removal forms a diamagnetic cavity with \( \Delta B/B_0 \sim 1 \). Anthropogenic examples of this type are the high-altitude nuclear explosions of the early 1960s,\(^1\) chemical releases in space,\(^2\) and magnetized laser-produced plasmas (MLPP).\(^3\) Research into these types of expansions has been focused on their instabilities,\(^4\) the diamagnetic cavity,\(^5\) shock waves,\(^6\) and their ability to radiate waves.\(^7\)–\(^9\) The latter two depend on the way in which the expansion can couple to an ambient plasma. Since the magnetic field can do no work on individual particles and these environments are all characterized by large ion-ion mean-free-paths, understanding the coupling of energy and momentum from an energetic expansion to a magnetized, ambient plasma is tied to understanding the total electric fields.

A popular model of such a collisionless interaction is to assume that the expansion speed is constant and that the electrostatic field is negligible. This leads in the case of a spherical expansion to a dipole-like magnetic field structure and coupling due to the resulting induced electric field. The collisionless interaction occurs via “Larmor” coupling where the ions gyrate in their accelerating field. This is a very general definition and subject to rather ambiguous application. In this article, it will specifically refer to the models of Refs. 10–13. In those models, Larmor coupling occurs via radially outward motion of the ambient ions as they gyrate either in or due to the acceleration of the induced electric field, which is in the azimuthal direction and largely external to the expansion. They are accompanied by strong assumptions regarding relevant time and spatial scales. For example, in Refs. 11–13, the expansion is modeled as a thin shell of uniformly expanding plasma (non-decelerating) with Alfvén Mach numbers much greater than 1, \( M_A \gg 1 \), so as to entirely neglect the electrostatic field. In the application to strong coupling, the thin shell moving at high speed is presumed to have fields strong enough to accelerate ambient ions to the expansion speed on a timescale comparable to the cyclotron period. References 12 and 13 go further and derive their oft-cited coupling parameter by extending Larmor coupling to infinite time despite the eventual stagnation of the plasma. This is not to say that Larmor coupling does not occur and that those models are very restrictive in their applicability. The Larmor coupling models do have the advantages of (1) providing relative simple solutions to the electromagnetic fields and (2) the production of structures that resemble and may lead to shocks.

This model, however, does not share much in common with experimental high-\( \beta \) expansions. First, no expansion into an ambient magnetic field nor one coupling energy to an ambient plasma can have a constant expansion speed. The reduction in speed from magnetic forces and coupling to the ambient plasma necessarily mean that the induced electric field weakens in time. Second, there is a potentially wide sheath with a substantial electrostatic field\(^15\) that can dwarf the induced electric field. This sheath depends on the distribution of electrons near the boundary of the expansion, which is not necessarily a thin shell. Finally, the compression in the magnetic field does not often exhibit a clear dipolar form and even includes a current reversal.\(^5\) This implies that the induced electric field external to the expansion may not have the same sign as the expanding dipole model.

In this article, we consider the collisionless coupling of a high-\( \beta \) expansion to an ambient plasma accounting for the former’s observed behavior. A heuristic model and its experimental basis are provided. The coupling is examined in enough detail to establish basic scaling relations in the weak coupling limit, that is, the limit in which the presence of the ambient plasma does not significantly affect the evolution of the expansion. The strong coupling limit is briefly discussed as a natural extension to the results of the weak coupling limit.

MODEL OF LAMINAR FIELDS

The phenomenology of energy exchange between the expanding plasma and its environment can be described using a model similar to that used by Lord Rayleigh in 1917...
(Ref. 16) to characterize gaseous bubble dynamics. The original paper concerned itself with the dynamics of a gas-fluid interface, denoted by a characteristic radius for the cavity trajectory $R(t)$, and resistance to its expansion through a pressure provided by the external fluid. The subsequent motion of the interface, in the ideal case, follows an oscillatory exchange of energy between kinetic energy of the expansion and potential energy corresponding to the work done by the external pressure. Realistically accounting for viscosity and radiation, the oscillatory motion eventually damps.

The formation of the diamagnetic cavity in MLPP experiments follows an almost identical qualitative description; one merely replaces the external fluid with the magnetic field and uses the component of the kinetic energy that is opposed by this pressure. The Rayleigh model resulting from energy conservation is thus

$$dE = 0 = \frac{C}{2} N_d m_d \left( V_{r0}^2 - (\partial R)^2 \right) - \frac{DB^2}{2\mu_0} (1 + \Gamma) \Delta Vol,$$

(1)

where $\Delta Vol$ is the change in the volume of the expansion; $V_{r0}$ is the initial radial velocity; $B_0$ is the ambient magnetic field; $\Gamma$ is any additional external pressures (e.g., the mass loading of the ambient plasma) normalized to the magnetic field pressure; $m_d$ is the mass of debris ions; $N_d$ is the number of debris ions; and $C$ and $D$ are the $O(1)$ profile-averaged values of cross-field kinetic energy and the magnetic field energy, respectively, and are assumed constant. Quantities of similar nature associated with the ambient and debris ions will here and after be distinguished by superscript or subscript “a” and “d,” respectively. For a spherical expansion, $\Delta Vol \sim R^3 - R_0^3$, where $R_0$ is the initial radius of the plasma. The quantity $CN_d m_d V_{r0}^2/2$ represents the initially available kinetic energy, $E_k$, of the expansion which is to be expended on expulsion of the magnetic field and work against external pressures. This quantity is some small fraction of the source energy, $E_s$, due to reflection, ionization, and radiation losses: $E_s = \eta E_{S0}, \eta < 1$. Normalizing the spatial variable to its initial value, $\tilde{R} = R/R_0$ and $\tau$ to the characteristic hydrodynamic time, $\tau = t/\tau_0$, where $\tau_0 \equiv R_0/V_{r0}^2$, and the resulting differential equation with $\Gamma = 0$ is

$$\partial_\tau \tilde{R} = \sqrt{1 - \frac{1}{\beta} (\tilde{R}^3 - 1)},$$

(2)

where $\beta$ is the ratio of the initial kinetic energy density to magnetic field energy density.

The solution to Eq. (2) for the initial condition $(\tilde{R}, \tau) = (1, 1)$ can be expressed implicitly in terms of an incomplete beta function or a hypergeometric function. For the present case of high-$\beta$ expansions, taking $\beta \gg 1$ allows an exact solution up to time $\tau_D = \tau_0 [1 + (1/3)\Gamma(1/2)\beta^{1/3}/(\Gamma(5/6)^3)]$, where the numerical coefficient of $\beta^{1/3}$ evaluates to $\approx 1.402$. This is the time of peak diamagnetism when the maximal amount of energy is in the magnetic field term in Eq. (1) and the expansion across the magnetic field stagnates. This time also corresponds to a second characteristic spatial dimension, $R_B = \beta^{1/3} R_0$, often called the magnetic bubble radius.

Renormalizing variables in Eq. (2) to $\tau_D$ and $R_B$ results in a curve independent of $\beta$ provided $\beta \gg 1$. This curve is shown in Fig. 1 along with experimental data that the author was able to extract from several high-$\beta$ expansion experiments that had sufficiently clear and complete $R(t)$ curves. From this figure, it is clear that a spherical Rayleigh model of high-$\beta$ expansions is very representative of the evolution of an expanding plasma and diamagnetic cavity.

The relationship between peak diamagnetism and the ambient magnetic field strength provided by the spherical Rayleigh model is also one of the most well-established in experiments on MLPPs. Both the values of $\tau_D$ and $R_B$ scale as $B_0^{-2/3}$. Figure 2 shows a selection of the excellent agreement between these quantities over wide variation in $B_0$. Note that the 4 points following the blue line, $\tau_D \sim B_0^{-2}$, were points for which the assumption $\beta \gg 1$ was likely broken and the authors of that paper also pointed to a degraded quality of confinement of their MLPP.

FIG. 1. Rayleigh model solution of Eq. (2) compared to previous experimental data from Refs. 3, 18, and 19 for MLPPs and Ref. 2 for AMPTE. Also shown are curves with $\Gamma$ = external pressure/magnetic field pressure and the lower bound determined via virial theorem.\(^{20}\)

FIG. 2. Experimental verification of the relationship between $\tau_D$, or equivalently $R_B$ [Eq. (2)], as a function of magnetic field for vacuum expansions. $V_n$ is normal to the target surface.
to exist. The most important of which are associated with thermal effects in the expansion. In particular, the kinetic energy term originates in the initial thermal content of the debris particles. An isolated plasma with thermal energy density sufficiently greater than external pressures will convert its thermal energy into directed motion. This conversion occurs on the order of the initial hydrodynamic time and results in a supersonic expansion.\textsuperscript{25} To ignore this process and other thermal processes which must also occur on this timescale, the conditions \( t \gg t_0 \) and \( R \gg R_0 \) must be satisfied. If one chooses \( t > 10t_0 \), and by extension \( R > 10R_0 \), as sufficient time and space to ignore this conversion process, this condition is equivalent to demanding \( \beta > 10 \) and is why the present considerations are restricted to high-\( \beta \) expansions.

The spherical model is not the only one that has been applied to such an expansion.\textsuperscript{26} One might expect the volume to behave more like an elliptical expansion due to the anisotropic effect of the magnetic field. This would result in a volume term in Eq. \( \text{(1)} \) such as \( \Delta \text{Vol} = (R^2 - R_0^2)\text{d}V_0 \), where \( V_0 \) is the velocity along the ambient magnetic field. The difference in \( R(t) \) caused by such a term only becomes noticeable near and after peak diamagnetism for values of \( V_0 \approx V_\phi \). The choice of a spherical expansion is one of the simplest and most used ones. Inclusion of the non-autonomous term complicates the analysis without adding much physical insight. A theory for the post-expansion phase \(( t > t_\phi )\) would undoubtedly have to account for this behavior, but that is beyond the scope of this paper.

The curve represented by Eq. \( \text{(2)} \) has been calculated numerous times but often in the context of estimating an effective ‘\( g \)‘ for use in fluid instability models.\textsuperscript{4,27} From measured electromagnetic fields,\textsuperscript{15} the slowing down is due to the presence of a Hall effect electrostatic field. It was found that if resistivity is neglected, the electric fields can be described by Ohm’s law of the form

\[
E = E_{\text{ind}} + E_{\text{es}} \approx -V \times B + \frac{1}{en_e}J \times B, \tag{3}
\]

where \( n_e \) is the time-dependent electron density and the right-hand-side constitutes an electrostatic field, \( E_{\text{es}} \), and induced electric field, \( E_{\text{ind}} \). In a cylindrically symmetric expansion \(( r, \phi, z )\) with \( B_0 = B_0 e_\phi \), \( V_0 \times B_0 \approx V_\phi B_0 e_\phi \), and \( J \approx J e_\phi \), the induced and electrostatic fields can be separated in the right-hand-side of Eq. \( \text{(3)} \) to yield

\[
E_{\text{ind}} \approx -V \times B, \quad E_{\text{es}} \approx \frac{1}{en_e}J \times B.
\]

The term \( V \times B \) is mostly an induced electric field, provided that there is no substantial rotation of the expanding plasma—a fact not yet experimentally observed and will not be expanded upon here. The simple interpretation is that the induced field keeps the electrons drifting with the ballistic ions across the magnetic field and the electrostatic field mediates energy exchange from radial ion motion to electron currents, eventually forming the diamagnetic cavity. Since the induced electric field is proportional to the expansion speed and the electrostatic field is responsible for deceleration, the values for \( \partial_R^2 \) and \( \partial^2_{R} \) from the solution to Eq. \( \text{(2)} \) can be substituted to get a characteristic magnitude of these fields, respectively. This gives the relations

\[
|E_{\text{ind}}| \approx \frac{m_i e V_0^2}{Z_{\text{de}} R B} \left( \frac{\omega_{ci}^2 \tau_D}{1.402} \right) d_R, \tag{4a}
\]

and

\[
|E_{\text{es}}| \approx \frac{m_i e}{Z_{\text{de}} R B} \left( \frac{R}{R_B} \right)^2, \tag{4b}
\]

where \( Z_{\text{de}} \) is the debris charge state and \( \omega_{ci}^2 \) is the cyclotron frequency corresponding to a debris ion moving in the ambient magnetic field. Equations \( \text{(4a)} \) and \( \text{(4b)} \) only provide characteristic strengths for the electric field components localized in a sheath region located at \( r = R(t) \). Accounting for realistic profiles for the local fluid velocity field, \( V(r, t) \), density, \( n(r, t) \), and magnetic field, \( B(r, t) \), will surely result in electric fields different from those above. The ratio and evolution of these characteristic fields are still useful as they provide ways to characterize the motion of ambient ions as they interact with the high-\( \beta \) expansion.

The ambient ions and debris ions respond to the same fields of the diamagnetic cavity modified only by their different charge to mass ratio. Thus, another important dimensionless quantity is the ratio of the charge to mass ratios which will be expressed as \( \omega_{ci}^2 / \omega_{ci}^2 \). Equivalently, one can consider the dimensionless quantity \( \omega_{ci}^2 \tau_D \). The quantity \( \omega_{ci}^2 \tau_D \) for both ambient and debris ions is commonly given in the literature in the numerically similar form of a ratio of the cavity radius to a directed Larmor radius, \( R_B / \rho_L := R_B \omega_{ci} / V_\phi \). It is important to note that this Larmor radius is not representative of the Larmor radius of any particle in the system. To ascribe a physically meaningful Larmor radius is to assume at least an approximate orbital motion in configuration space and either an initial velocity condition or a known characteristic velocity. The first of these assumptions is not guaranteed in rapidly varying fields with \( DB / B_0 \sim 1 \), while the second is easily misapplied. The expression \( \omega_{ci}^2 \tau_D \), on the other hand, is the maximal change in the phase of an ion in velocity space when subject to a given magnetic field strength. This interpretation does not itself depend on any initial condition of a particle nor necessarily on any orbital motion of the particles, but it is directly connected to the changes from the initial conditions. Since the magnetic field is time-dependent and drops to 0 in an ideal cavity, the actual change in the phase, or fraction of a gyroperiod, is almost certainly less than \( \omega_{ci}^2 \tau_D \). A model where the ambient ions gyrate in response to the electric fields, and hence move orthogonal to them, would therefore require \( \omega_{ci}^2 \tau_D \gg 1 \). Conversely, the response for \( \omega_{ci}^2 \tau_D < 1 \) is mostly directed along the electric fields, and the magnetic field itself plays a minor role. This separates the ambient ion response into four qualitatively different momentum coupling scenarios for a high-\( \beta \) expansion. These are summarized in Table I. Essentially, \( \omega_{ci}^2 \tau_D \) sets the type of field responsible for coupling during the expansion and \( \omega_{ci}^2 \tau_D \) sets the ultimate direction of the momentum coupling.
TABLE I. Laminar Collisionless Coupling Scenarios. Qualitative overview.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Coupling Field</th>
<th>Response of ambient ions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 &lt; \omega^2_{ci} \tau_D &lt; 1$</td>
<td>$-\nabla \phi$</td>
<td>$\Delta V_e \approx \Delta V_s^i$, implosive impulse</td>
</tr>
<tr>
<td>$1 \leq \omega^2_{ci} \tau_D &lt; 0.3$</td>
<td>$-\beta A$</td>
<td>$\Delta V_e \approx \Delta V_s^e$, explosive gyration</td>
</tr>
<tr>
<td>$0 &lt; \omega^2_{ci} \tau_D &lt; 1$</td>
<td>$-\nabla \phi$</td>
<td>$\Delta V_e \approx \Delta V_s^i$, vortical gyration</td>
</tr>
<tr>
<td>$0 &lt; \omega^2_{ci} \tau_D &lt; 1$</td>
<td>$-\beta A$</td>
<td>$\Delta V_e \approx \Delta V_s^e$, vortical impulse</td>
</tr>
</tbody>
</table>

Substituting the solution to Eq. (2) into Eqs. (4a) and (4b) along with $\omega^2_{ci} \tau_D = 0.3$ yields the curves in Fig. 3 for the time dependent scales of the electric field components. These curves emphasize that at early times, the induced electric field dominates but monotonically decreases up to $t = \tau_D$. Conversely, the electrostatic field monotonically increases in strength as the expansion loses energy to the magnetic field. The ratio of the peak strengths is determined by the parameter $\omega^2_{ci} \tau_D$, which for most MLFP experiments has been on the order of 1 or even smaller. See Table II for an extensive, but not exhaustive, list of typical experimental values of the Rayleigh model parameters. The dynamics described by Eq. (4) within the spherical Rayleigh model show the importance of the electrostatic field in such contexts. There is always some time for which $|E_{ei}| > |E_{im}|$, and due to the spherical geometry, the expansion covers larger volumes of ambient material at later times than at earlier times.

After peak diamagnetism, the induced electric field must reverse with the collapse of the diamagnetic cavity, implying that the ultimate coupling is strongly affected by the electrostatic field. This would be in stark contrast to popular models of coupling which employ uniformly expanding superconducting spheres or neglect the electrostatic field entirely.1,11–13 Only for expansions such as Starfish prime, where $\omega^2_{ci} \tau_D \gg 1$ and the induced field dominates for a sufficiently long time, is it possible that the Larmor coupling models might apply.

**WEAK COUPLING**

To estimate the response of the ambient plasmas in the weak-coupling limit, we assume that the only electric fields are those described by the Rayleigh model [Eq. (4)] and are spatially well-represented by spherically symmetric, self-similar functions. Other contributions to the electric fields—e.g., finite resistivity, finite electron temperature, and instability—are ignored as “fringe” fields. Under this assumption, the electric field as viewed by a particle at a radial position $r$ can be written as

$$E' = \frac{E}{E_0} = x_\phi \phi \partial_t f \left( \frac{r}{R} \right) \hat{r} \hat{\phi} + \frac{3}{2} R^2 \gamma \left( \frac{r'}{R} \right) \hat{r},$$

where $E_0 = m_0 V_0^2 / Z q R B_0$, $x_\phi = \omega_{ci} \tau_D / 1.402$, $t' = t / \tau_D$, $r' = r / R_B$, and $R = R(t) / R_B$. The functions $f(x)$ and $g(x)$ are the self-similar induced and electrostatic field profiles, respectively. To simplify the particle motions, we assume that the particles do not move significantly relative to the electric fields while they are being accelerated. The impulse from such a field is

$$\Delta V_a(r,t) = \frac{Z e}{m_p} \int_{R(t)=r}^{R(t)=r'} dt' E' \approx 1.402 \beta \int_{r'=R(t)=r'}^{R(t)=r} dt' E'(r', t'),$$

which is proportional to the time spent by the particle passing through the electric field. In the above equation and what follows, subscripted $t$ variables are integration variables and primed time variables are normalized to $\tau_D$. The approximation that the particle does not move during the acceleration is the same as requiring $\Delta V_a \cdot e, \partial \phi / \partial R \sim V_0$. The above expression satisfies this condition for either the case of a narrow sheath, i.e., $f(x)$, $g(x)$ are substantially different from 0 only outside of a narrow region $1 \gg \delta x$(sheath width) near $x = 1$, or the case $\omega^2_{ci} \tau_D^2 < 1$.

The total energy coupled to a uniform, ambient plasma when the cavity has radius $R(t)$ is

$$E_c(t) = 4 \pi m_n n_i \int_0^{R(t)} dr \left( \Delta V_a(r,t) \right)^2 = 8 \pi m_n n_i R_B^3 \left( 1.402 \beta \int_{r'=R(t)=r'}^{R(t)=r} dt' E'(r', t') \right)^2 I(t'),$$

where

$$I(t') = \int_0^{t'} dt'_1 \int_0^{t'_1} dt'_2 \int_0^{R(t'_2)} d\tau E_\phi(r', t'_1) E_\phi(r', t'_2) + E_\phi(r', t'_1) E_\phi(r', t'_2).$$

This form allows for easy calculation of the energy coupled once a self-similar profile for the electric fields is supplied.

To get an idea of the relative strength of the coupling, the energy coupled to the ambient plasma can be treated as an additional external pressure acting upon the expansion. For Eq. (1), one would arrive at

$$\Gamma = \frac{p_{ext}}{p_B} = \frac{E_c(t)}{4 \pi R^3 DB_0} = \frac{6(1.402)^2 \beta x_\phi}{C} \int_0^{R_B} \left( \frac{R_B}{R} \right)^3 \left( \frac{R_B}{R} \right)^3 I(t'),$$

TABLE II. Typical parameters in high-$\beta$ expansions.

<table>
<thead>
<tr>
<th>Reference</th>
<th>$E_s$ (J)</th>
<th>$B_0$ (G)</th>
<th>$\beta$</th>
<th>$\omega_p^3 \tau_D$</th>
<th>$\omega_p^3 \tau_D$</th>
<th>$R_m/R_B$</th>
<th>$R_q/R_B$</th>
<th>$\omega_p^3 R_B^3$</th>
<th>$\omega_p^3 R_B^3$</th>
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<td>$5 \times 10^4$</td>
<td>4</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>29$^b$</td>
<td>30</td>
<td>1000</td>
<td>$3 \times 10^6$</td>
<td>0.2</td>
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<td>3</td>
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<tr>
<td>5</td>
<td>1</td>
<td>600</td>
<td>$10^6$</td>
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<td>0.5</td>
<td>40</td>
<td>30</td>
<td>0.02</td>
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</tr>
<tr>
<td>15</td>
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<td>750</td>
<td>$7 \times 10^5$</td>
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<td>0.04</td>
<td>1</td>
<td>6</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>30$^{b,c}$</td>
<td>6</td>
<td>700</td>
<td>$7 \times 10^5$</td>
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<td>0.1</td>
<td>0.2</td>
<td>0.2</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>20</td>
<td>250</td>
<td>$6 \times 10^6$</td>
<td>0.6</td>
<td>2</td>
<td>0.3</td>
<td>0.1</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>32$^b$</td>
<td>20</td>
<td>600</td>
<td>$6 \times 10^6$</td>
<td>0.2</td>
<td>0.1</td>
<td>0.02</td>
<td>0.03</td>
<td>50</td>
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<tr>
<td>33$^b$</td>
<td>100</td>
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<td>$2 \times 10^5$</td>
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<td>3</td>
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<td>0.3</td>
<td>5</td>
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<tr>
<td>35</td>
<td>150</td>
<td>710</td>
<td>$5 \times 10^6$</td>
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<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>0.5</td>
</tr>
<tr>
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<td>0.9</td>
<td>0.007</td>
<td>0.009</td>
<td>100</td>
<td></td>
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<tr>
<td>AMPTE$^2$ (3/21/85)</td>
<td>$\sim 10^{15}$</td>
<td>$8 \times 10^{-4}$</td>
<td>$\gg 10^9$</td>
<td>1.2</td>
<td>160</td>
<td>$2 \times 10^9$</td>
<td>150</td>
<td>0.9</td>
<td></td>
</tr>
<tr>
<td>Starfish Prime$^{1c}$</td>
<td>$\sim 10^{15}$</td>
<td>$30$</td>
<td>$10^3$</td>
<td>0.3</td>
<td>$10^3$</td>
<td>40</td>
<td>230</td>
<td>0.03</td>
<td>10$^3$</td>
</tr>
</tbody>
</table>

$^a$Not all $E_s$ is relevant to $\beta$. To calculate $\beta$, a conversion efficiency of $\epsilon = 0.1 = E_s/E_b$ from laser energy to kinetic energy was used, which is conservative for inverse bremsstrahlung-dominated coupling of the laser to the target. An alternative way to calculate $\beta$ is to use the definition $\beta = (R_q/R_B)^3$ if $R_B$ is given explicitly or is visible in the data. $R_B$ is the effective radius of the focal point on the target calculated assuming uniform illumination of a circle with the experimental laser characteristics.

$^b$Experiments with expansions directed perpendicularly often only quote the velocity on the blow-off axis, which is a mix of the cavity expansion velocity and the center-of-mass velocity. The marked experiments do not have sufficient data to account for this. This potentially means an underestimate of the critical parameters $\tau_D$ and $(R_q/R_B)^3$.

$^c$Conducted in a non-uniform ambient plasma. Only a characteristic $R_m/R_B$ is given.

where $p_B$ is the magnetic field pressure, $C$ and $D$ are as in Eq. (1), and $R_q$ is called the equal charge radius defined as

$$R_q = \frac{3 \ Z_d \ N_d \ a_i}{4 \ \pi \ Z_d \ a_{ii}}.$$

It is clear from Eq. (6) that an important parameter for the coupling strength is $\omega_p^3 R_m^3 / \omega_p^3 R_q^3$, which is given in Table II for high-$\beta$ expansion experiments and will be referred to as the “coupling parameter.” It is a measure of the amount of energy imparted to the ambient ions by electric fields associated with the diamagnetic cavity. It is proportional to the initial energy of the expansion, the volume or size of the expansion, and the density of the ambient plasma. In the form given above, it shows clearly the effect of relative “electromagnetic inertia” of the ambient and debris ions subject to the same field, $\omega_p^3 / \omega_p^3$.

Another quantity that appears frequently in the literature is the equal mass radius

$$R_m^3 = \frac{3 \ m_a \ N_{ai} \ a_i}{4 \ \pi \ m_a \ a_{ii}}.$$

Clearly, $R_q$ and $R_m$ are related by $R_q^3 = \omega_p^3 R_m^3 / \omega_p^3$. Furthermore, $R_m$ can be related to the ambient Alfvén Mach number, $R_m^3 / R_B^3 = M_A^{-3}$. The last expression makes $R_m$ an appealing parameter for the generation of shocks. In the context of similarity scaling, only two of the three new dimensionless variables mentioned above involving the ambient plasma—$\omega_p^3 \tau_D$, $R_q/R_B$, and $R_m/R_B$—are independent with the last and the Alfvén Mach number, $M_A$, being degenerate. Similarity tells us the choice of which two are used is a matter of convenience, while the other can be completely removed from the analysis. Table I and Eq. (6) suggest that the simplest and most physically relevant choice is $\omega_p^3 \tau_D$ and $R_q/R_B$ because together they describe how the ambient plasma reacts and how much it might feedback onto the expansion. The absolute magnitude of $\Gamma$ and this feedback depends ultimately on $I$.

From Eq. (5), it is found that $I(\tau)$ strongly depends on and increases with the width of the sheath layer at the boundary of the diamagnetic cavity, specifically the time an ion spends within the sheath. Since the time an ambient ion spends in the sheath region and the sheath width are all smaller than 1 in the normalized variables, it is easy to see that $I(\tau)$ will be much less than 1 without very intense fields. As an example calculation, consider the case of a cold, steady expansion of gas into vacuum. The standard application of Rayleigh’s model in this case uses a velocity profile, $V_r \sim r \partial_r R/R$. Continuity then gives a density variation $n(r, t) \sim R^{-3}$. This density variation was shown to be a very good approximation for unmagnetized laser-produced plasma expansions. Neglecting ion pressure and viscous effects, the Rayleigh model yields the momentum conservation equation

$$m_0 \ \frac{r}{R} \partial_t^2 R = J \times B \ / \ n.$$

It is identical to that given in Ref. 16 if the ambient pressure is replaced by a magnetic pressure. For a uniform density profile, this in turn leads to the instructive, albeit unrealistic profile $B_c = B_0 \Gamma / R$. The corresponding self-similar electric field profiles are $f(\chi) = x^3$ and $g(\chi) = x$. Inserting these into Eq. (5) and separating the contributions to $I$ of the induced and electrostatic fields.
\[ I_\phi'(r) = \frac{\omega_{ci}^4}{5 \cdot 6 \cdot 7} R^2(\tau) \]

and

\[ I_r'(r) = \frac{9}{4 \cdot 5} \int_0^{\tau_r} dt_1 \int_0^{R^2(t_1)} dt_2 R^3(t_2). \]

respectively. For \( \omega_{ci}^4 \tau_D = 0.3 \) and the solution to Eq. (2), one arrives at \( I_\phi(1) = 0.0002 \) and \( I_r(1) = 0.029 \). As expected, the case of \( \omega_{ci}^4 \tau_D < 1 \) corresponds to coupling dominated by the electrostatic field. We can further estimate how large of a coupling parameter is allowed while still being in the weak-coupling regime. From Eq. (6), we see that \( \omega_{ci}^4 R_B^3 / \omega_{ci}^4 R_q^3 < 10 \) is sufficient for these profiles to yield \( \Gamma < 1 \) and weak coupling. Note that more realistic profiles that exhibit a narrower sheath, which would be represented as higher order polynomials in \( x \) near \( x = 1 \), would result in larger powers of \( R' \) in the above expressions. The time integrals in \( I_r \) would be smaller as the particles spend less time in narrower sheaths. As for \( I_\phi \), the assumption that \( \Delta v_r / \partial_R \ll 1 \) in order to produced instantaneous acceleration breaks down for the induced field before \( \tau_D \). If acceleration ceases prior to \( \tau' = 1 \), the integrals in \( I_\phi \) decrease with larger powers of \( R' \). Thus, narrower sheaths than the above example would result in less efficient coupling overall.

**Strong coupling**

The case of strong coupling can only be qualitatively described by the Rayleigh model; a quantitative analysis would require a full simulation. Some obvious effects of strong coupling are that the ambient plasma modifies the magnetic field profile and the electric fields, which would vary through the parameters \( R_m \) and \( R_q \). A systematic experimental study of this has not been done and would be quite difficult due to the weak dependence of the Rayleigh model on experimental value parameters. From the terms in Eq. (3), two effects can be surmised: charge loading and mass loading. For mass loading, whereby the ambient ion mass increases thus decreasing \( R_m / R_B \) or increasing \( M_A \), the Lorentz term, \( V \times B \), is apparently suppressed as the additional mass reduces the mass-averaged velocity. Similarly for charge loading, an increasing ambient ion charge, or decreasing \( R_q / R_B \), will suppress the Hall term, \( J \times B / n_e \). Because of the relationship between \( R_m \) and \( R_q \) and the effect of ambient mass on Eq. (3), one might expect that the overall energy coupled to the ambient plasma decreases with increasing \( M_A \) and the balance between the Lorentz and Hall terms determined by \( \omega_{ci}^4 / \omega_{ci}^4 \) in the strong coupling limit.

The way in which the trajectory \( R(t) \) is modified for strong coupling can be easily estimated by assuming that the energy coupled depends on time only through linear volumetric expansion. That is, \( E_r(t) \sim R^3 E_r(\tau_D) / R_p^2 \). Introducing this into Eqs. (6) and (1), \( \beta \) is effectively reduced by a factor \( 1 + \Gamma \). The resulting solutions to the modified Eq. (2) for various \( \Gamma \) values are shown in Fig. 1. Also shown is the virial analysis by O’Neill,\(^{20} \) which shows the solution in the case of maximal shock loading or instability of the expansion. This simple relationship allows much of the discussion of the momentum coupling in the weak coupling limit to be carried over to the strong coupling limit by replacing \( R_B \) with the reduced diamagnetic cavity size \( R_D = R_B / (1 + \Gamma)^{1/3} \) and similarly reduced \( \tau_D \). This shows that a significant deviation of \( R(t) \) requires the magnetic pressure and the effective pressure from the collisionless ambient-debris interaction to be at least comparable. In other words, the electromagnetic field must impart a greater energy to the ambient ions than it keeps for itself.

**DISCUSSION AND CONCLUSION**

Several features of the coupling parameter of the Rayleigh Model are more attractive than those derived from assumptions of Larmor coupling\(^{12,13,38} \) and empirical modifications of these from shock simulations.\(^{39} \) First, the coupling parameter of the Rayleigh model increases with ambient density. Coupling parameters from Larmor coupling tend to increase with \( R_m \) or \( R_q \), suggesting that energy transfer increases with decreasing density. The coupling parameter from the Rayleigh model is more intuitive and more analogous with hydrodynamics, whereby increasing the ambient mass decreases the energy retained by the debris. This does not necessarily mean that Larmor coupling is wrong but rather points out that care should be taken in the applicable range of coupling parameters: Larmor coupling parameters are often derived by assuming that strong coupling is present, or sufficiently high ambient density, whereas the Rayleigh model is more accurate in the limit of vanishing ambient density. Second is the volumetric relationship itself. Physically, one would expect that the expansion dynamics should depend on \( R^d \), where \( d \) is the dimensionality of the expansion. Third is that the coupling parameter of the Rayleigh model properly accounts for the relative “electromagnetic inertia” of the debris-ambient interaction. That is, the parameter \( \omega_{ci}^4 R_B^3 / \omega_{ci}^4 \) is a measure of the relative response of the ambient and debris particles to the electromagnetic fields to which both species must be subject. Finally, because of its equivalent form, \( \left( \omega_{ci}^4 / \omega_{ci}^4 \right) M^2_A \), this coupling parameter increases with increasing Alfvén Mach number. This is attractive for shock studies with \( M_A > 1 \), which require a transition from weak coupling to strong coupling between the expansion and the ambient plasma. Although the Rayleigh model was derived specifically in the limit of weak coupling, it may be expanded as mentioned above with an appropriate, non-zero \( \Gamma \) while keeping in mind that any coupling parameter that increases with \( R_m \) or \( R_q \) decreases with increasing \( M_A \).

For interest in generating shocks with a high-\( \beta \) expansion, it is important to note that the dependence on the Mach number as given here is more related to a geometric feature of the expansion than it is a statement that a discontinuous fluid structure exists. In fact, the derivation of the coupling parameter made no assumptions on the internal structure of the electromagnetic fields. It only demanded that they have certain scaled strengths so that they are consistent with energy conservation and experimental observations. To account for the structure of the various fields is to solve the more complicated problem of local momentum conservation. Within the Rayleigh model, this is equivalent to ascribing a form for the function \( I(t) \), which may acquire a dependence
on any of the parameters \( \omega_{a1}^D, \omega_{a2}^D, \beta, R_B/R_a \), or any other dimensionless parameters associated with additional physics such as dissipation mechanisms.

A feature that has not been addressed is the effect of a common geometry in MLPP experiments. Many are conducted such that the majority of debris is directed unilaterally away from the target and across the background magnetic field.\(^{5,6,28–35} \) This configuration includes the presence of a polarization field, which arises from the center-of-mass motion of the expanding plasma. If the center-of-mass moves with velocity \( V_m \), then the polarization field has the approximate strength \( E_{pol} = -V_m \times B \). If both \( V_m \) and \( B_0 \) are irrotational, as is the case for a uniform magnetic field and uniform translational motion of the plasma, this polarization field is purely electrostatic in nature. If not, there is no simple classification of \( E_{pol} \) as induced or electrostatic. This is the case during the main expansion phase of the diamagnetic cavity. Therefore, there is no easy way to account for its presence as it destroys the symmetry of the total electric field by superposing itself on the spherical cavity fields in the Rayleigh model.

If we neglect fringe and surface fields, the polarization field persists long after the cavity has decayed since the system has no net force opposing uniform translational motion. Thus, the cavity may proceed as a spherical expansion moving at the faster speed \( V_m + \partial_R R \) along the expansion direction but still \( \partial_B R(t) \) across it, where \( R(t) \) is the solution to Eq. (2). This is consistent with experiments and was alluded to in the footnote of Table II. After \( t_D \) when \( \partial_B R = 0 \), the polarization field continues coupling more energy to the ambient plasma with or without the continuing presence of the diamagnetic cavity decays. This may inflate or even dominate the plasma with or without the continuing presence of the diamagnetic cavity. This may inflate or even dominate the plasma with or without the continuing presence of the diamagnetic cavity. This is the case during the main expansion phase of the Rayleigh model.

We have examined the collisionless coupling of a high-\( \beta \) expansion to an ambient plasma in the context of a Rayleigh-type, spherical expansion model, which is popular and successful for gaseous expansions in fluids. This model provides estimates of the electric field scale strengths in terms of the expansion parameters. The parameter \( \omega_{a1}^D \) determines the relative strength of the induced electric field to the electrostatic field. The parameter \( \omega_{a2}^D/\omega_{a1}^D \) or equivalently \( \omega_{a2}^D t_D \), then determines how the ambient plasma responds to those electric fields. The traditional model of Larmor coupling associated with shocks produced by superconducting expansions corresponds to the limit \( \omega_{a2}^D t_D, \omega_{a2}^D t_D \gg 1 \) which few, if any, laboratory experiments have attained for an axisymmetric expansion. The amount of energy transferred to a plasma of massive particles scales as \( \omega_{a2}^DR_B^3/\omega_{a1}^DR_a^3 \), which is necessary, but not sufficient, to have greater than 1 to see strong coupling effects.

**ACKNOWLEDGMENTS**

Support for this work was provided by the U.S. Department of Energy and the National Science Foundation.

Primary funding was provided through the Department of Energy Grant No. DE-SC0001605.

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20. L. Rayleigh, Philos. Mag., Ser. 6 34(200), 94 (1917).