Non-linear Alfvén wave interaction leading to resonant excitation of an acoustic mode in the laboratory

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The nonlinear three-wave interaction process at the heart of the parametric decay process is studied by launching counter-propagating Alfvén waves from antennas placed at either end of the Large Plasma Device \cite{Gekelman1991}. A resonance in the beat wave response produced by the two launched Alfvén waves is observed and is identified as a damped ion acoustic mode based on the measured dispersion relation. Other properties of the interaction including the spatial profile of the beat mode and response amplitude are also consistent with theoretical predictions for a three-wave interaction driven by a nonlinear ponderomotive force. A simple damped, driven oscillator model making use of the MHD equations well-predicts most of the observations, but the width of the resonance curve is still under investigation. C 2015 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4919275]

I. INTRODUCTION

Alfvén waves, a fundamental mode of magnetized plasmas, are ubiquitous in plasmas in the laboratory and in space. While the linear behavior of these waves has been extensively studied,\textsuperscript{1–5} nonlinear effects are important in many real systems, including the solar wind and solar corona. In particular, a parametric decay process in which a large amplitude Alfvén wave decays into an ion acoustic wave and backward propagating Alfvén wave may play a key role in establishing the spectrum of solar wind turbulence.\textsuperscript{6} Ion acoustic waves have been observed in the heliosphere, but their origin and role have not yet been determined.\textsuperscript{7} Such waves produced by parametric decay in the corona could contribute to coronal heating.\textsuperscript{8} Parametric decay has also been suggested as an intermediate instability mediating the observed turbulent cascade of Alfvén waves to small spatial scales.\textsuperscript{8,9}

A large-amplitude Alfvén wave propagating along the background magnetic field is unstable against decay into a counter-propagating Alfvén wave and an ion acoustic mode.\textsuperscript{10–17} To satisfy frequency and wave number matching relations, the counter-propagating wave must be downshifted from the pump frequency by the frequency of the acoustic mode. In an MHD treatment of the problem with vanishing plasma beta, the sound wave must propagate in the direction of the pump Alfvén wave for instability to occur.\textsuperscript{10} The instability grows through the coupling of finite density perturbations to the nonlinear $\vec{v} \times \vec{B}$ force on the ions. This force is due to a cross product between the fields of the pump and daughter Alfvén waves; thus, the growth rate is proportional to the pump amplitude. A kinetic treatment of the interaction by Hasegawa and Chen\textsuperscript{11} found that the growth rate could be significantly enhanced for large perpendicular wave number.

To date, there has been an abundance of theoretical work,\textsuperscript{10–12,14–17} but very little direct experimental observation of the parametric decay of large amplitude Alfvén waves. Observations by Spangler et al.\textsuperscript{18} in the ion foreshock region upstream of the bow shock in the Earth’s magnetosphere indicate the presence of large amplitude Alfvén waves as well as density fluctuations with no magnetic spectral component; the latter are presumed to be acoustic modes resulting from parametric decay. However, the pump wave lifetime is often short compared to the instability growth rate; in this case no acoustic waves are observed. Furthermore, $\beta \sim 1$ in these plasmas; in this regime, there are significant departures from fluid theory as ion kinetic effects become important.\textsuperscript{19} A later set of observations by Narita et al.\textsuperscript{20} also found only indirect evidence for wave-wave interactions in the foreshock region. Therefore, a well-controlled and well-diagnosed laboratory experiment would be a valuable tool to validate simple theoretical predictions and aid in comparison with observational evidence.

The Large Plasma Device (LAPD) at UCLA is an ideal environment for this experiment. Extensive prior work has focused on the properties of linear Alfvén waves.\textsuperscript{5,21–23} Studies of the nonlinear properties of Alfvén waves have also been performed on LAPD.\textsuperscript{24,25} For instance, the interaction between two co-propagating Alfvén waves may non-resonantly drive a quasimode in the plasma; this quasimode then couples to the original waves to produce a spectrum of Alfvénic sidebands.\textsuperscript{24} In a second set of experiments, the co-propagating beat mode is driven near the frequency of existing drift wave turbulence; the beat mode is then found to couple to the drift modes, altering the drift wave frequency spectrum.\textsuperscript{25}

In this paper, the first laboratory observations of the Alfvén-acoustic mode coupling at the heart of the parametric decay instability described in Dorfman and Carter\textsuperscript{26} are presented in detail. Counter-propagating shear Alfvén waves are launched from antennas and allowed to interact nonlinearly. As the beat frequency between these two launched waves is...
varied between discharges, a resonant response is observed when frequency and wave number matching is satisfied for coupling to an ion acoustic mode. Other features of the interaction including the beat mode spatial structure and response amplitude match predictions based on a three-wave interaction driven by a nonlinear ponderomotive force. Note, however, that these results represent a beat wave process rather than an instability; direct observation of parametric decay in the lab remains the focus of ongoing experiments.

The paper is organized as follows: Sec. II describes the experimental setup in the LAPD. Key results of the study including the resonant response and beat mode dispersion relation are covered in Sec. III. A simple MHD model for the beat response along with possible explanations of the observed damping are discussed in Sec. IV. Finally, concluding points are presented in Sec. V.

II. EXPERIMENTAL SETUP

The LAPD at UCLA, shown in schematic form in Figs. 1 and 2, is an ideal environment for experiments diagnosing nonlinear Alfvén wave interactions. The LAPD is an 18 m long cylindrical vacuum vessel capable of producing a 16.5 m long, 60 cm diameter, quiescent, magnetized plasma column for wave studies. The BaO cathode discharge plasma lasts for ~10 ms, including a several ms-long current flattop.

FIG. 1. Schematic of the LAPD. On the left is a cutaway view of the device showing the magnets (yellow and purple) which produce a relatively uniform axial magnetic field $B_z$. To the right, a picture of a typical Helium discharge (top) and another view of the plasma source (bottom). Reprinted with permission from Phys. Plasmas 18, 055501 (2011). Copyright 2011 AIP Publishing LLC.

FIG. 2. Experimental setup in the LAPD plasma column. Alfvén wave antennas shown at either end of the device launch the counter-propagating Alfvén waves to be examined in this study. Magnetic probes B4 and B7 and Langmuir probes L1 and L2 placed between the two antennas in the plasma column are used to diagnose the interaction; various $z$ positions are used for the data in the paper. Reprinted with permission from S. Dorfman and T. A. Carter, Phys. Rev. Lett. 110, 195001 (2013). Copyright 2013 The American Physical Society.

A typical current waveform is shown in Panel (A) of Fig. 3. Because the mesh anode and cathode are located within 55 cm of each other at one end of the device, most of the plasma column carries no net current. The magnetic field coils shown in Fig. 1 span the length of the device, allowing for relatively uniform axial magnetic fields ($B_z$) between 0.4 and 2 kG. Typical densities and temperatures are $n_e \sim 10^{12} /\text{cm}^3$ and $T_e \sim 5 \text{ eV}$ ($\beta \ll 1$); a fill gas of helium or hydrogen was used for the present studies. Example plasma parameters are summarized in Table I. There are 450 ports located at various places along the vessel that allow for easy insertion of various magnetic and electrostatic diagnostics.

For the present set of experiments, loop antennas placed at either end of the LAPD, shown in Fig. 2 launch linearly

TABLE I. LAPD plasma parameters measured for two of the setups used in the present experiments. These example parameters were used to produce two of the points in Fig. 7.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$\text{He}^+$</th>
<th>$\text{H}^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density, $n$ ($\times 10^{12} \text{ cm}^{-3}$)</td>
<td>1.6 ± 0.18</td>
<td>1.2 ± 0.16</td>
</tr>
<tr>
<td>Electron temperature, $T_e$ (eV)</td>
<td>4.3 ± 0.9</td>
<td>4.3 ± 1.0</td>
</tr>
<tr>
<td>Magnetic field, $B_0$ (G)</td>
<td>750</td>
<td>450</td>
</tr>
<tr>
<td>$v_A = B/\sqrt{4\pi n_m}$ (km/s)</td>
<td>648 ± 37</td>
<td>898 ± 60</td>
</tr>
<tr>
<td>$C_R = \sqrt{T_e/m_e}$ (km/s)</td>
<td>20 ± 2</td>
<td>20 ± 2</td>
</tr>
<tr>
<td>$v_{th} = \sqrt{2T_e/n_e}$ (km/s)</td>
<td>1236 ± 132</td>
<td>1225 ± 136</td>
</tr>
<tr>
<td>$f_\delta = eB/2\pi m_e c$ (kHz)</td>
<td>286</td>
<td>686</td>
</tr>
<tr>
<td>$f_\omega = n_e (dr/dt)$ (kHz)</td>
<td>12 ± 1</td>
<td>38 ± 3</td>
</tr>
<tr>
<td>$f_{\text{aux}} \equiv \frac{6\pi n_e v_e^3 \log(A)}{m_e^{3/2}}$ (MHz)</td>
<td>6 ± 2</td>
<td>4 ± 2</td>
</tr>
<tr>
<td>$f_{\text{Alfven}}$ (kHz)</td>
<td>220</td>
<td>480</td>
</tr>
<tr>
<td>$\rho_s = C_s/\Omega$ (cm)</td>
<td>0.57 ± 0.06</td>
<td>0.47 ± 0.05</td>
</tr>
<tr>
<td>$\beta_i = \frac{8\pi n_i T_i}{B_i^2}$</td>
<td>$(5 \pm 1) \times 10^{-4}$</td>
<td>$(10 \pm 3) \times 10^{-4}$</td>
</tr>
</tbody>
</table>
polarized, counter-propagating Alfvén waves with amplitudes of $\delta B \sim 1$ G during the discharge-current-flattop period of the LAPD discharge. Typical timing of the antenna signals with respect to the discharge current is illustrated by Fig. 3. Each antenna consists of an insulated copper wire loop; currents along the background axial magnetic field produce perpendicular wave magnetic field, launching the mode.39 The antenna is driven using modular high power MOSFET push-pull drivers. Capacitors placed in parallel with the antenna inductance form an $RC$ resonant circuit that allows a monochromatic Alfvén wave to be driven when appropriately tuned.

It should be noted that parametric decay of a single Alfvén wave is not observed in these experiments: consistent with this, the experimental value of $\delta B/B_0 \leq 2 \times 10^{-3}$ gives a growth rate10 that is comparable to an Alfvén wave transit time through the entire plasma column for relevant experimental parameters. Instead, antennas directly launch both the “pump” and “daughter” Alfvén waves at similar amplitudes. In the plasma column between the antennas, magnetic probes detect the magnetic field signatures of the launched waves while Langmuir probes are used to detect signatures of a density response at the beat frequency. Because the LAPD plasma has excellent shot-to-shot reproducibility, experiments may be performed by repeatedly running the same discharge parameters while scanning a moveable probe. A novel computer-programmable ball flange port allows this to happen quickly on a shot-by-shot basis, which means that data can be acquired continuously. This approach may be used to construct a 2-D profile in the $x$-$y$ plane averaged across multiple discharges.

III. RESULTS

A clear nonlinear response at the beat frequency is observed in these experiments, as shown in Fig. 4. When the two Alfvén wave antennas are turned on between 8 ms and 10 ms in this helium discharge, a beat wave at the difference frequency of 14 kHz is observed both in the filtered ion saturation current trace displayed in Panel (A) and the full frequency spectrum shown in Panel (C). This signal will be shown to have many properties consistent with an ion acoustic mode produced by a three-wave matching process. The beat amplitude of 75 $\mu$A represents $\sim 3.5\%$ of the measured mean ion saturation current. After the antennas are turned off and the last of the magnetic signatures from the Alfvén waves pass by a fixed magnetic probe at $t = 10.015$ ms, the amplitude of the beat wave does not immediately drop to zero, indicating that coupling to a normal mode of the plasma has occurred. The ring-down time of the driven wave is $\sim 85$ $\mu$s, comparable to an ion-neutral collision time of $\sim 100$ $\mu$s for these parameters. When these experiments are repeated in hydrogen plasmas, the ring down time is shorter; consistent with this, the ion-neutral collision frequency for the chosen parameters is larger in hydrogen than it is in helium.30

The same non-linear response is also observed by a microwave interferometer measuring line integrated plasma density. This is illustrated in Fig. 5. During the time period the antennas are on, the filtered interferometer signal shown in Panel (A) indicates the presence of density fluctuations at the difference frequency. As with the ion saturation current signal in Fig. 4, the fluctuations ring down after the pump Alfvén waves turn off, indicating coupling to a normal mode in the plasma.

The beat amplitude is expected to be largest when three-wave coupling most efficiently excites a normal mode of the plasma. The experimental strategy to test this prediction is as follows: the launch frequency of the cathode side antenna is held fixed while the launch frequency of the end mesh side antenna is varied between discharges. The plasma response at the beat frequency is then examined in each discharge to find the difference frequency that best couples to an acoustic...
The results of this scan for helium plasmas with $B_0 = 750$ G are represented by the dashed line in Fig. 6. This curve, representing the beat wave amplitude with both antennas on plotted as a function of the difference frequency, peaks at a frequency of around 13 kHz. This frequency at which three-wave matching relations are best satisfied to excite a normal mode of the plasma is defined as the resonance frequency.

A calculation based on the ion acoustic and Alfvén wave dispersion relations allows for a prediction of the observed resonance frequency. For the measured experimental parameters, $V_A/v_{th,i} \approx 1$, suggesting that a kinetic calculation of the dispersion relation would be appropriate. However, because the collisionality is fairly high ($\nu_{\text{mfp},e} \sim 0.2$ m, $k_i\nu_{\text{mfp},e} < 1$) and in order to keep the calculation simple, a fluid dispersion relation is used for kinetic Alfvén waves$^{21}$ (KAWs): \[ \omega = k_{||}V_A \sqrt{1 + (k_{\perp}\rho_i)^2 - \left(\omega_c/\Omega_i\right)^2}. \]

The relevant dispersion relation for the ion acoustic mode is \[ \Delta\omega = \Delta k_{||}C_s \sqrt{1 + (\Delta k_{\perp}\rho_i)^2}. \]

Three-wave matching relations predict that $\Delta \omega = \omega_2 - \omega_1$, $\Delta k_{||} = k_{||2} + k_{||1}$, and $\Delta k_{\perp} = k_{\perp2} - k_{\perp1}$ where 1 and 2 are subscripts associated with the counter-propagating Alfvén waves. The plus sign in the $\Delta k_{\perp}$ equation comes from the geometry of the two overlapping Alfvén wave cones. Some simple algebra and the assumptions $\omega_1 \approx \omega_2 \approx \omega > \Delta \omega$, $k_{\perp1} \approx k_{\perp2} \equiv k_{\perp}$ lead to the equation

\[ \Delta \omega = \sqrt{\frac{\omega \sqrt{2}\rho_i}{1 + (k_{\perp}\rho_i)^2 - \left(\omega_c/\Omega_i\right)^2}}, \]

Plugging in the experimental parameters used to produce Fig. 6, including a typical $\beta_i \sim 8 \times 10^{-4}$, Eq. (1) predicts a resonant frequency of 13 kHz. This agrees well with the experimental result.

Equation (1) is satisfied for a wide range of plasma and antenna parameters; this is shown in Fig. 7. For fixed ion mass, Eq. (1) implies that the resonant frequency is a function of $\omega_c/\Omega_i$. Therefore, magnetic field scans and scans of the main antenna frequency may be overplotted on Fig. 7. The resulting datapoints for each gas fall within the gray shaded region calculated using Eq. (1). The finite width of the gray region represents the statistical uncertainty from an average over similar Langmuir probe measurements. Temperature is measured by sweeping the voltage of a single tip; density is obtained from the measured temperature and ion saturation current measurements.

The frequency and wave number matching relations used to derive Eq. (1) hold over a broad axial range. This is verified in Fig. 8 which shows the measured phases of the two launched Alfvén waves and the beat acoustic mode over 4.2 m of the 9.0 m between the antennas. When the measurements for each mode are fit to a line, each Alfvén mode is found to have a $k_{||}$ of about 3/m while the acoustic beat mode $k_{||}$ is the sum of the two at 6/m. Furthermore, the direction of the acoustic mode $k_{||}$ is shown to be in the direction of the higher frequency launched Alfvén mode, consistent with theoretical predictions based on momentum conservation.$^{10}$

The spatial profile of the beat response suggests a ponderomotive drive mechanism. This is shown experimentally in Fig. 9. The measured wave magnetic field vectors are plotted as white arrows; overlapping current channels for the two Alfvén waves are indicated by the circulation pattern of

![FIG. 6. Beat amplitude as a function of beat frequency $\Delta f$ showing a resonant response in helium plasma with background $B_0 = 750$ G. The Alfvén wave antenna on the cathode end produces a fixed frequency 230 kHz wave while the frequency of the wave produced by the end mesh antenna is scanned from 205 to 230 kHz between discharges. The dashed curve shows the beat amplitude $\delta\omega$ as a percent of $\omega$ observed as a function of $\Delta f$ for a Langmuir probe at $z = 6.07$ m. The amplitude is normalized to the zero frequency component; the shaded errorbar represents the level of background fluctuations. The thin dashed-dotted trace is the predicted density response $|\rho_i/\rho_0|$ based on Eq. (3) using $N = 0.14$, $\beta_i = 8 \times 10^{-4}$ and $b_i/B_0 = 1/750$. The thin solid trace is the equivalent prediction for $N = 0.36$. The figure inset to the right shows the $N = 0.36$ trace rescaled to the data to emphasise that this $N$ is the best fit for the width of the peak. Reprinted with permission from S. Dorfman and T. A. Carter, Phys. Rev. Lett. 110, 195001 (2013). Copyright 2013 The American Physical Society.]
For ion acoustic modes based on the dispersion relation. For nism as will be discussed in Sec. IV.

the color scale, the beat amplitude is greatest near the origin which is where the Alfvén wave magnetic field peaks. This observation is consistent with a ponderomotive drive mechanism as will be discussed in Sec. IV.

The resonance in the beat wave response is identified as an ion acoustic mode based on the dispersion relation. For each of the experimental runs in Fig. 7, the parallel wave number for the response at the resonance frequency is determined by examining the phase delay between two Langmuir probes closely spaced in z (One such experimental setup is shown in Fig. 2). The result, shown in Fig. 10, is a linear dispersion relation with phase speed comparable to the sound speed for LAPD parameters. Using a kinetic dispersion relation\(^\text{11}\) for ion acoustic modes and assuming an ion temperature of 1 eV (previously measured in both helium and argon\(^\text{23}\)), the phase speed in helium requires \(T_e = 6.7 \pm 0.4\) eV, well in line with the value of 4.3 \(\pm 0.9\) eV measured. Since ion temperature has not been measured in hydrogen, one possible explanation for this discrepancy is \(T_i > 1\) eV. This could be a combination of higher background ion temperature and enhanced ion heating by the launched Alfvén waves in hydrogen plasmas, due to the lighter ion mass. Because, unlike helium, hydrogen is a molecular gas, the Franck-Condon effect may also lead to increased translational energy when molecular hydrogen is excited during dissociation.\(^\text{32}\) If the ion temperature is 2 eV, the ion acoustic mode dispersion relation\(^\text{11}\) requires only \(T_e = 5.2 \pm 0.3\) eV which is within errorbar of the measured value.

IV. DISCUSSION

To gain insight into the width of the response curve in Fig. 6, it is useful to model the nonlinear interaction as a damped, driven oscillator. In the simplest possible model, the ion acoustic perturbation is considered to be much smaller than the background density such that \(\rho_i/\rho_b \ll 1\) and \(\rho_i/\rho_b \ll \nu_i/\nu_b\). Self-consistent with this approximation, the parallel ion velocity perturbation is much smaller than the phase speed of the driven mode by the same order.

![FIG. 8. \(k_z\) matching over a broad axial range. Measured phases from probes at various \(z\) locations show an approximately linear phase change for the two Alfvén waves and the acoustic beat mode. The later is shown to propagate in the direction of the higher frequency Alfvén mode, consistent with theoretical predictions.](image)

![FIG. 9. Spatial profile of the beat and Alfvén waves in hydrogen plasma. On the left plot, white vectors represent the superimposed magnetic fields from the two Alfvén waves while the color scale shows spatial variations of the beat amplitude. The cathode and end mesh side antennas are set to 480 kHz and 450 kHz, respectively. \(B_0 = 450\) G. Magnetic data is averaged over the times at which the antenna-launched waves are in phase such that the magnetic field amplitude is at a maximum. The two current channels of the Alfvén waves are visible in the upper right and lower left. In the right plot, the measured magnetic field amplitude and beat response are shown as a function of \(R\), the distance from the origin along the diagonal cut shown. The beat amplitude is peaked near the origin where the local magnetic field peaks. Reprinted with permission from S. Dorfman and T. A. Carter, Phys. Rev. Lett. 110, 195001 (2013). Copyright 2013 The American Physical Society.](image)

![FIG. 10. Dispersion relation of beat waves at the resonance frequency. Experimental runs are the same as in Fig. 7. The best fit dashed line represents a phase speed of 29.1 \(\pm 0.7\) km/s in the hydrogen plot and 12.5 \(\pm 0.3\) km/s in the helium plot. Reprinted with permission from S. Dorfman and T. A. Carter, Phys. Rev. Lett. 110, 195001 (2013). Copyright 2013 The American Physical Society.](image)
Combining the MHD momentum and continuity equations and neglecting perpendicular propagating effects
\[
\frac{\partial^2 \rho}{\partial t^2} + \nu_{in} \frac{\partial \rho}{\partial t} + C_s^2 \frac{\partial^2 \rho}{\partial z^2} = \frac{\partial^2}{\partial z^2} \left[ \frac{b_{\perp 1} \cdot b_{\perp 2}}{B_0} \right]. \tag{2}
\]

Equation (2) describes a damped, driven oscillator system. The first and third terms represent the wave equation for an ion acoustic mode, the second term describes damping due to ion-neutral collisions, and the fourth term is the nonlinear ponderomotive drive that results from interaction between the two Alfvén waves. This acoustic mode drive term accelerates ions parallel to \( B_0 \) through a nonlinear \( \dot{v} \times \dot{B} \) force in the parallel ion momentum equation. For the ordering assumptions used to derive Eq. (2) to hold, the amplitude of the Alfvén wave drive at resonance must be small, \( b_{\perp 1}/B_0 \ll N_e/\sqrt{\beta_e}/2 \), where \( N = \nu_{in}/\omega_0 \) represents the collisionality normalized to the resonance frequency. During the time both Alfvén waves are turned on, the system will respond at the drive frequency \( \omega_D = \omega_2 - \omega_1 \) and the drive wave number \( k_D = k_{\parallel 2} + k_{\parallel 1} \). The response function at \( \omega = \omega_D \) and \( k = k_D \) follows from the linearization of Eq. (2):
\[
\left[ \begin{array}{c}
\rho_1 \\
\rho_0
\end{array} \right] = \frac{2 \left[ b_{\perp 1}, b_{\perp 2} \right]}{B_0} \frac{1}{\sqrt{(1 - \Omega_D^2)^2 + N^2 \Omega_D^2}},
\tag{3}
\]
where \( \Omega_D = \omega_D/\omega_0 \) is the drive frequency normalized to the natural resonance of the system \( \omega_0 = k_D C_s \). Qualitatively, Eq. (3) agrees well with the overlapping magnetic field and ion saturation current profiles shown in Fig. 9. Also consistent with this MHD theory, a scan of the antenna power (Fig. 4, panel (D)) reveals that the beat-driven amplitude grows proportionally to the product of the two Alfvén wave amplitudes.

Equation (3) is overplotted on Fig. 6 for \( N = 0.14 \) (thin dashed-dotted line) and \( N = 0.36 \) (thin solid line). While the amplitude of the resonant response is well predicted by \( N = 0.14 \), the scaled inset figure shows that the best fit for the width of the peak is obtained for \( N = 0.36 \). Furthermore, \( N = 0.36 \) is more than double the value \( N = 0.14 \) obtained from the ringdown time in Fig. 4. Despite this discrepancy, the \( N = 0.14 \) result implies that the ponderomotive force is of sufficient amplitude to drive the observed resonant response.

Note that the simplest possible model presented here contains a number of assumptions that do not precisely hold in the LAPD. For example, the plasma is assumed to be uniform and the finite \( k_{\perp} \) and spatial damping of the launched Alfvén waves is ignored. In the remainder of this discussion section, these three possibilities are analyzed in detail. However, it is found that neither variation of the plasma density nor the finite damping and \( k \) spectrum of the antenna can completely explain the observations. Other possibilities considered include the inclusion of the Hall term in the model as well as ion-electron collisionality. However, both of these cancel when the fluid equations are combined, leading to no net change to Eq. (2). Despite this open issue, the simple model presented here explains most of the observations well.

### A. Variation of plasma parameters

One possibility for the broader peak observed is the variation of plasma parameters as a function of location in the experiment. Acoustic beat modes measured by the Langmuir probe at a given location may have propagated from a nearby region that has slightly different values of temperature and density. Since the initial amplitude of a beat mode will depend on the plasma parameters at the location at which it is generated, a more rigorous formula for the response curve will result from a Green’s function solution to Eq. (2).

The solution will be constructed under the assumption that the variation of background parameters does not affect the form of Eq. (2). This means that while density may be a function of space, there is little variation of the electron temperature. Therefore, the Alfvén speed is a function of space while the acoustic speed is constant. This means that when two-counter-propagating Alfvén waves are launched, the wave frequencies \( \omega_1 \) and \( \omega_2 \) are a constant function of space while the values of \( k_1 \) and \( k_2 \) will be spatially dependent. To construct the response under these conditions, first consider the impulse response \( g(z, s) e^{-i\omega_D t} \) which results when the drive term on the right side of Eq. (2) is replaced by \( \delta(z - s) e^{-i\omega_D t} \):
\[
g(z, s > z) = -\frac{1}{\mathcal{R}(\Delta k_0) C_s^2} e^{i\Delta k_0 (s - z)} \left[ e^{i\Delta k_0 (s - z)} + \Delta k_0 + \frac{\nu_{in}}{C_s^2} \right]. \tag{4a}
\]

\[
\Delta k_0 = \frac{\omega_D}{C_s^2} \sqrt{1 + \frac{\nu_{in}}{\omega_D}} \tag{4b}
\]

For \( s < z \), \( g(s, z) \) is identically zero. In other words, the wave is propagating from the delta function source at \( z = s \) in the negative \( z \) direction. This propagation direction is chosen for consistency with the experimental data.

Note that the delta function driving term contains all wave numbers in equal measure. The resonance of Eq. (2), defined by \( \Delta k_0 \) in Eq. (4b), produces the largest response. Since the drive frequency \( \omega_D \) is set by the frequency difference between the two launched modes, and the collisionality \( \nu_{in} \) depends on neutral density and ion temperature, Eq. (4b) predicts that \( \Delta k_0 \) will have no plasma density dependence and hence no axial variation. However, axial variation is introduced in the response though the actual driving term. From Eq. (2) this term is
\[
f(z) = \mathcal{R} \left( \frac{b_{\perp 1} \cdot b_{\perp 2}}{4\pi} \frac{\partial^2}{\partial z^2} e^{-i\omega_D t - i\phi(z)} \right). \tag{5}
\]

The term \( \phi(z) \) in Eq. (5) represents the advance in phase due to the driving wave number. For a uniform plasma, \( \phi(z) = k_D z \), but if there is axial density variation, \( k_D \) will also be a function of \( z \). Keeping the general phase variation \( \phi(z) \), the response function is given by
\[
\frac{\rho}{\rho_0} = \frac{1}{\rho_0} \int_{-\infty}^{\infty} g(z, s) f(s) ds
\]
\[
\frac{\rho}{\rho_0} = \frac{2}{\beta_e} \left[ \frac{b_{\perp 1} \cdot b_{\perp 2}}{B_0^2} \right] \left[ -\cos(\phi(z) + \omega_D t) + \Delta k_0^2 \mathcal{R}(\Delta k_0) \int_{-\infty}^{\infty} e^{i\Delta k_0 (s - z)} \cos(\phi(s) + \omega_D t) ds \right]. \tag{6}
\]
The integral on the right hand side of Eq. (6) is in the form of a convolution of two sinusoidal signals. This will be largest when both signals have the same wave number argument; in other words, when the wave number of the local spatial variation \( \phi(z) \) best matches the resonant wave number \( \Re(\Delta k_0) \), the response amplitude will peak. The integral therefore represents a weighted average of this matching over the ion acoustic mode source region for a given measurement location. For the portion of the source region far from the measurement location, the contribution will be attenuated due to the factor of \( \Im(\Delta k_0) \) in the exponent.

In order to construct a response function from Eq. (6), it is necessary to pick a specific form for the axial density variation which will lead to an expression for \( \phi(z) \). For example, for the simplest case \( \phi(z) = k_D z \), Eq. (3) is recovered. For more complicated cases, the response function must be numerically integrated. One such case is a linear density variation \( \phi(z) = k_D (1 + a k_D z) \) where \( k_D \) and \( a = (1/k_D)(1/n) (\partial n/\partial z) \) are constants. For \( a > 0 \), density is highest near the cathode (large \( z \)) and decreases in the direction of the wave vector of the acoustic mode. From Eq. (1), this means that locations in the source region will have a higher resonance frequency than the measurement location. Consistent with this the peak labeled “Ndec” is shifted to the right in Fig. 11.

By the same argument, the peak labeled “Ninc” is shifted to the left. In the case of the curve labeled “Sin,” the variation in density is sinusoidal. The two peaks in this curve represent contributions from the high and low points in the sinusoid where density variation is minimal. A similar double-peaked response curve has been observed in some LAPD measurements and thus may indicate the presence of sinusoidally varying quantities.

Comparing the model response curves in Fig. 11 to the “R0” curve with no density variation, it is clear that axial variation in the plasma parameters can lead to a broader response curve. However, out of the three broader curves tried in Fig. 11, the “Ninc” curve has a shape that is most similar to the data. But this translates to density increasing in the direction away from the plasma source in the experiment and is therefore unlikely. Detailed axial density measurements may help to match the shape of the curve, although the analysis here reveals that this is unlikely.

### B. Spatial damping of the pump waves

Because the pump Alfvén waves are damped by ion-electron collisions, \( b_{1,1} \) and \( b_{1,2} \) in Eq. (2) will be an exponentially decaying component in addition to a sinusoidal component. For the Hydrogen case reported in Fig. 8, an examination of the pump mode structure as a function of \( z \) reveals that the ratio of the imaginary to the real part of the parallel wave number is about 0.1. However, because the two Alfvén waves are launched from opposite ends of the device and have similar damping rates, the product \( b_{1,1} b_{1,2} \) remains approximately constant with \( z \) over the measured range. Equivalently, the imaginary component of \( k_D \) is negligible because it is equal to the difference of damping rates between counter-propagating waves of similar characteristics. Therefore, the spatial damping of the pump waves does not have a significant effect on the response curve and cannot be responsible for the broadening.

### C. Finite k spectrum of the pump waves

A third possibility for the broader response curves observed is that a spectrum of \( k_\parallel \) and \( k_\perp \) is launched by each antenna, resulting in a slightly different resonance frequency for each spectral component. When the response curves for each component are added together, the result may be broader that what is found using the simple model in Fig. 6.

This situation may be modeled using the Green’s function response from Eq. (4a) with a driving term that takes into account various combinations of parallel wave numbers \( k_D j \)

\[
\rho(z) = \Re \left( \sum_j \frac{b_{1,1j} \cdot b_{1,2j}}{4\pi} \beta_j^2 \frac{e^{-\omega_D t - ik_D z}}{C_2} \right). 
\]

The associated response function is

\[
\frac{\rho}{\rho_0} = \sum_j \left[ b_{1,1j} \cdot b_{1,2j} \right] \frac{k_j}{(k_j - 1)^2 + k_j^2 (k_j + 1)^2 + \Gamma^2} \times \left[ 1 - k_j^2 + \Gamma^2 \right] \cos(k_D z + \omega_D t) 
+ 2\kappa_j \Gamma \sin(k_D z + \omega_D t) ,
\]

where the dimensionless ratios \( \kappa_j = k_D j / \Re(\Delta k_0) \) and \( \Gamma = \Im(\Delta k_0) / \Re(\Delta k_0) \).

An analysis of Eq. (8) shows that a typical \( k \) spectrum does not provide sufficient broadening to explain the experimental results. Based on the KAW dispersion relation and the measured \( k_\perp \) spectrum from a Bessel function decomposition of a typical antenna pattern, the values of \( k_D j \) and the associated amplitudes may be inferred. These values are then plugged into Eq. (8) and the magnitude \( |\rho/\rho_0| \) is shown as a function of beat frequency in Fig. 12. Two cases are shown: the solid red line represents a single value of \( k \) while the dashed green line represents the results from a typical measured \( k_\perp \) spectrum. While including a finite \( k \) spectrum

![Graph](image-url)
FIG. 12. Comparison of Eq. (8) to experimental data. The response curve obtained from Eq. (8) is shown for a single $k_\perp$ component as well as for a typical $k_\parallel$ spectrum. Note that these curves are re-scaled to the peak amplitude to allow for a comparison of the widths. While including a range of wave numbers does lead to a broadening of the curve, this is not sufficient to explain the observations.

does broaden the response curve, the broadening is not sufficient to explain the observations.

V. CONCLUSIONS

In summary, the first laboratory observations of the Alfvén-acoustic mode coupling at the heart of parametric decay are presented. Counter-propagating Alfvén waves launched from either end of the LAPD produce a resonant response identified as an ion acoustic mode based on the dispersion relation, spatial profile, and other features consistent with a simple MHD theory. Several areas for further investigation remain. Ion acoustic waves have never been directly launched by an antenna in the laboratory at densities comparable to those in the LAPD. Thus, a new technique to directly launch ion acoustic waves is being developed and will be utilized for a detailed study of the damping mechanism. The new technique will also be used to investigate Alfvén and ion acoustic wave coupling; the launched acoustic mode could potentially seed the parametric decay process. Additional studies may also focus on parametric decay from a single large amplitude Alfvén wave; simulations are underway to determine the best possible parameter regime for this experiment.

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27See http://plasma.physics.ucla.edu/bapsf for Basic plasma science facility at ucla.