Three-dimensional two-fluid Braginskii simulations of the large plasma device

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The Large Plasma Device (LAPD) is modeled using the 3D Global Braginskii Solver code. Comparisons to experimental measurements are made in the low-bias regime in which there is an intrinsic $E \times B$ rotation of the plasma. In the simulations, this rotation is caused primarily by sheath effects and may be a likely mechanism for the intrinsic rotation seen in LAPD. Simulations show strong qualitative agreement with the data, particularly the radial dependence of the density fluctuations, cross-correlation lengths, radial flux dependence outside of the cathode edge, and camera imagery. Kelvin Helmholtz (KH) turbulence at relatively large scales is the dominant driver of cross-field transport in these simulations with smaller-scale drift waves and sheath modes playing a secondary role. Plasma holes and blobs arising from KH vortices in the simulations are consistent with the scale sizes and overall appearance of those in LAPD camera images. The addition of ion-neutral collisions in the simulations at previously theorized values reduces the radial particle flux by about a factor of two, from values that are somewhat larger than the experimentally measured flux to values that are somewhat lower than the measurements. This reduction is due to a modest stabilizing contribution of the collisions on the KH-modes driving the turbulent transport. © 2015 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4931090]

I. INTRODUCTION

The work presented here builds upon an initial numerical study1 of the Large Plasma Device (LAPD)2 using the Global Braginskii Solver code (GBS).3 The LAPD creates a linear plasma approximately 17 m in length and 30 cm in radius with an axially-directed magnetic field. Under normal unbiased operating conditions considered here, the plasma exhibits an intrinsic, self-generated $E \times B$ rotation about the machine axis. It is suggested that this intrinsic rotation is due to sheath effects, which are seen to cause the rotation in simulations, although this has yet to be definitively shown in experiments. In the simulations, the spontaneously generated radial electric field is associated with an axisymmetric, positive, radially-decreasing plasma potential that counteracts the preferential parallel streaming of electrons to the end walls due to their larger thermal velocities. This preferential streaming charges the plasma positive until the net flow of ions and electrons to the end walls is roughly equal and thus $\langle J_{\parallel}\rangle \simeq 0$, where the brackets denote an average over time. The physics of this process leads to the well-known Bohm sheath boundary conditions

$$V_{\parallel i} = \pm c_s, \quad (1)$$

$$V_{\parallel e} = \pm c_s \exp (\Lambda - e\{\phi_{\text{plasma}} - \phi_{\text{wall}}\}/T_e), \quad (2)$$

where $c_s = \sqrt{T_e/m_i}$ is the ion sound speed, $\Lambda = \ln \sqrt{m_e/(2\pi m_i)} \sim 3$ is the sheath parameter for a nominal He plasma, $\phi_{\text{plasma}}$ is the plasma potential, and $\phi_{\text{wall}}$ is an optional biasing profile applied to the end-walls. A body of work4–8 has explored the impact of various biasing profiles on the plasma. For example, the side walls can be biased relative to the anode,4 leading to strong flows near the far edge of the device. Here, as a first step, we focus on the simplest case without external biasing or edge flows ($\phi_{\text{wall}} = 0$) using the configuration and parameters for the unbiased case presented by Carter and Maggs.5 In that limit, the tendency for the electron and ion fluxes to balance so that $V_{\parallel e} = V_{\parallel i}$ produces a positive plasma potential $\phi_{\text{plasma}} \sim \Lambda T_e$ that, like the temperature profile, decreases with radius. The temperature profile, as well as the plasma density, reflects a balance between the input of plasma and heat by the sources in the center of the device, the turbulent cross-field radial transport due to plasma instabilities, and the parallel loss of the plasma to the end walls at the ion sound speed. As found in some previous work,1 the main driver of the cross-field transport in our simulations is Kelvin-Helmholtz (KH) instabilities excited by the shear in the poloidal $E \times B$ flow. Resistive drift waves are also present, as are shear modes, at substantially smaller scales. Here, we go beyond the past work of Ref. 1 to focus on a detailed simulation-experiment comparison that includes a wide range of diagnostics. This work also takes a step beyond previous modeling efforts9–12 based on the Boundary Turbulence code (BOUT)13 that did not include either shear effects or spontaneous rotation. As a result, the main driver of transport in our simulations—KH modes generated by the sheath-driven equilibrium $E \times B$ shear—is absent in those studies.

II. MODEL EQUATIONS AND NUMERICAL SETUP

Our simulations are based on a modified version of the 3D GBS code which has been used to study turbulence and...
transport in a wide range of devices. Its uses include an initial study of LAPD, the simple magnetized torus, TORPEX, and a recent study focusing on the scrape-off layer (SOL) region. The code evolves a set of electrostatic two-fluid drift-reduced Braginskii equations that assume \( T_i \ll T_e \):

\[
\frac{dn}{dt} = -\frac{\partial (nV_{le})}{\partial z} + D_n(n) + S_n, \tag{3}
\]

\[
\frac{d\omega}{dt} = -V_{le} \frac{\partial \omega}{\partial z} + \frac{m_e e_B}{e n} \frac{\partial \Omega_i}{\partial z} - \nu_m \omega + S_\omega, \tag{4}
\]

\[
m_e \frac{dV_{le}}{dt} = -m_e V_{le} \frac{\partial V_{le}}{\partial z} + \frac{4}{3} \frac{n_e}{n} \frac{\partial^2 V_{le}}{\partial z^2} + e_B \frac{\partial \phi}{\partial z} + \frac{e}{n} \frac{\partial \phi}{\partial t} - T_e \frac{\partial T_e}{\partial z} - \frac{17}{3} \frac{\partial T_e}{\partial t}, \tag{5}
\]

\[
m_e \frac{dV_{le}}{dt} = -m_e V_{le} \frac{\partial V_{le}}{\partial z} + \frac{4}{3} \frac{n_e}{n} \frac{\partial^2 V_{le}}{\partial z^2} - \frac{1}{3} \frac{\partial \rho_c}{\partial z}, \tag{6}
\]

\[
\frac{dT_e}{dt} = -V_{le} \frac{\partial T_e}{\partial z} + 2\frac{T_e}{3} \frac{\partial T_e}{\partial t} - \frac{2}{3} \frac{T_e}{V_{le}} \frac{\partial V_{le}}{\partial z} + \frac{\partial T_e}{\partial t} (T_e) + \frac{\partial T_e}{\partial t} (T_e) + S_T, \tag{7}
\]

where \( \omega = V_{le}^{\perp} \phi \) is the vorticity, \( \rho_c = n_T \), \( A, B = \partial_t A, \partial_t B \), \( -\partial_t A, \partial_t B, dJ_2/dt = \partial f/\partial t - (c/B) \partial f/\partial z \), \( j = \text{en} (V_{le} - V_{le}) \), \( \Omega_i = eB/m_c, \nu_m \) is the ion-neutral collision rate, and \( n_{e0} \) and \( n_{t0} \) are the parallel ion and electron viscosities, respectively. We apply Bohm sheath boundary conditions (Eqs. (1) and (2) with \( \phi_{\text{wall}} = 0 \) on the outflows of ions and electrons to the end walls, for example, in the parallel current term in the vorticity equation and elsewhere. Sources for the density, temperature, and vorticity are represented, respectively, as \( S_n, S_T, \) and \( S_\omega \) and are described in further detail in Appendix A. The vorticity source, as explained in Appendix, represents the injection of high energy electrons due to the LAPD’s cathode source. For typical LAPD discharges, in which the plasma is pulsed on the order of 1 Hz with each pulse lasting about 10 ms and discharging a current on the order of 3–6 kA, the vorticity source has only a weak effect on the simulations, as discussed later.

Equations (3)–(7) are solved on a field-aligned Cartesian grid parallel to \( z \) using a finite difference method with fourth-order Runge-Kutta time stepping and small numerical diffusion terms that use the parallel and perpendicular diffusion operators, \( D \perp \) and \( D \). For this study, a grid size of \( n_x = n_y = 256 \) and \( n_z = 64 \) was used. The code normalization is based on nominal values for a He plasma with reference parameters: \( T_{e0} \approx 6 \text{eV}, \ n_e \approx 2 \times 10^{17} \text{ cm}^{-3}, \ \Omega_i = eB/m_c \approx 960 \text{ kHz}, \ c_{s0} = \sqrt{T_{s0}/m_i \approx 1.2 \times 10^6 \text{ cm/s}}, \) and \( \rho_{Do} = \rho_{Do}/\Omega_i \approx 1.4 \text{ cm}. \)

The side walls in the simulations, perpendicular to the magnetic field, span from \( -L/2 \) to \( L/2 \), where \( L = 100 \rho_{Do} \) is the domain width. For convenience, this width has been made somewhat larger than the actual LAPD diameter so that essentially all the plasma is lost in the parallel direction before reaching the side walls of the simulations. The parallel domain spans from \( -L/2 \) to \( L/2 \), where \( L = 36 \text{R} \) and is normalized to the LAPD radius, \( R = 0.5 \text{ m}. \)

Scales are normalized to the reference ion-sound gyroradius, \( \rho_{Do} \), and parallel scales to the machine radius \( R \).

The density and electron temperature are sourced using top-hat shaped profiles, \( S_n \) and \( S_T \), respectively, which mimic the injection front from the cathode source as shown in Fig. 1(a)

\[
S_n = S_{n0} \delta \left\{ 1 - \tanh \left( \frac{(r - r_s)}{L_s} \right) \right\} \exp \left( -\lambda_s z \right). \tag{8}
\]

Here, \( S_{n0} = 0.05 \rho_{Do} / R \) and \( S_{Do} = 0.05 \rho_{Do} c_{s0} / R \) are constants that adjust the strength of the source and are chosen to approximately match the source rates in the experiment, \( r_s = 20 \rho_{Do} \) is the radius of the cathode, \( \lambda_s = 0.015 / R \) an axial decay rate, and \( L_s = 0.5 \rho_{Do} \) a length scale that controls the steepness of the top-hat source. The exponential term is a simple model for collisions that mimics the weak drop-off in density and temperature seen axially in LAPD. All source parameters are held fixed throughout the simulations and thus the relaxation of the density and temperature profiles across the magnetic field is due to the onset of plasma instabilities as the simulations progress in time. Changes to GBS were also made to model ion-neutral collisions. In LAPD, neutral collisions are difficult to measure and are inferred through secondary measurements and theoretical estimates. As we show for convenience in Appendix B, ion-neutral collisions lead to a damping term in the vorticity equation that is proportional to the Pedersen conductivity. This damping weakens the KH-driven cross-field transport and thereby leads to a moderate steepening of the temperature, density, and potential profiles discussed later.

III. EVOLUTION OF TURBULENCE AND TRANSPORT

At the start of a simulation, plasma density and temperature are sourced down the length of the simulation domain as shown in Figure 2(a). As discussed earlier, the Bohm sheath...
boundary conditions (Eqs. (1) and (2)) on the outflows of ions and electrons to the end walls lead to an approximate global balance $V_{||} \sim V_{\perp}$, or

$$\omega_{\text{plasma}} \sim \Lambda T_e^2. \quad (9)$$

This balance is not forced upon the simulation, but arises spontaneously, in particular, from the tendency for the system to relax into a nearly time-steady, axisymmetric equilibrium state in which the parallel current term in the vorticity equation (like all of the other terms) is small. Figure 3 shows a radial cut of the potential in a simulation that has reached a quasi-steady turbulent state and its relationship to $\Lambda T_e$. Outside of the cathode’s ionization front (the region containing fast electrons accelerated from the cathode to anode that source plasma in LAPD), the plasma approximately follows the condition in Eq. (9) so that $\omega_{\text{plasma}} \sim 3T_e$. A modest deviation from this condition occurs within the ionization front of the cathode where fast electrons from the source give rise to a parallel current and thus a vorticity source, as explained in Appendix A. This inward injection of electrons off-sets the outward loss of electrons through the sheaths and thus leads to an effectively smaller value of $\Lambda$. This reduction in the effective $\Lambda$ can be seen in Figure 4 in which the averaged potential and temperature at a given radius have been plotted against one another to show how the sheath parameter changes leading up to the cathode edge. A constant slope of $\Lambda = 2$ is presented as a reference to the reader to show the effective reduction in the potential versus temperature curve inside the cathode edge. A more accurate relationship for Eq. (9) which accounts for the vorticity source, $S_\text{vort}$, and the Pedersen collisional term with $v_{\text{Ped}}$, is given in Appendix C.

The time averaged Reynolds stress in the vorticity equation can also be included in this calculation, but plays a negligible role in determining the equilibrium electric potential, being more than two orders of magnitude smaller than the main sheath terms. The Reynolds stress may be found to play a more significant role in biased runs of LAPD, but for the unbiased case presented in this work, its effect is overshadowed by the sheath terms in setting up the plasma potential.

We return to Fig. 1, which shows the progression of a mid-plane cut of the density profile from early to late times. A cut of the temperature profile looks similar, as does a cut of the plasma potential, which follows the temperature according to $\phi \approx \Lambda T_e$. The density, temperature, and plasma potential profiles, prior to the onset of instabilities, grow in response to the top-hat-like density and temperature source terms, producing steep gradients in these quantities that reflect the source profiles (e.g., Fig. 1(a)). After a short time, as shown in Fig. 1(b), resistive driftwave instabilities appear with $k_\parallel \rho_{\parallel} \approx 0.5$, where $k_\parallel$ is the azimuthal wavenumber and $\rho_{\parallel}$ is the local ion-sound gyroradius at $r \approx 20 \rho_{\parallel}$ where the fluctuations are strongest. The wavenumber ($k_\parallel \rho_{\parallel} \approx 0.5$) is calculated from a poloidal cut at $r \approx 20 \rho_{\parallel}$, unwrapping and linearly detrending the result, and applying a peak finding algorithm, as shown in Figure 5. The measured value $k_\parallel \rho_{\parallel} \approx 0.5$ is close to that of the fastest growing driftwave mode predicted by standard resistive drift wave theory. This theory predicts a maximum growth rate for $k_\parallel \rho_{\parallel} \approx 0.6$ given by $\gamma \approx 0.085 (1 + 1.71 \eta) c_s \ell_\parallel / L_n \eta$ with $\eta = L_n / L_T$, and $L_n$ and $L_T$ are the respective gradient scale lengths for the density and temperature. This maximum growth rate is reached for $k_\parallel \approx 0.24 \sqrt{\nu / c_s \ell_\parallel}$, where $\nu = e^2 n_0 R / (m_r e \sigma_n)$ is the parallel resistivity. For typical LAPD parameters, this
maximally-unstable $k_{||}$ is comparable to the machine length. Simulations at larger values of the electron resistivity, $\nu$, have been carried out to verify the expected scaling $k_{||} \propto \sqrt{\nu}$.

Despite their fast onset, the drift waves saturate at such small amplitudes that, at later times, they are hard to discern in the later, nearly $k_{||} \approx 0$ turbulent state shown in Fig. 2. As the temperature continues to steepen and the associated $E \times B$ shear flow increases, Kelvin Helmholtz waves are eventually driven unstable and soon dominate the system (Fig. 1(c)). Previous work by Rogers and Ricci in the low-bias case showed that the subsequent cross field transport is driven mainly by these linearly unstable, $k_{||} \approx 0$, relatively long perpendicular-wavelength Kelvin Helmholtz modes.

Figure 6 shows three snapshots of time-averaged density profiles as plasma is sourced down the machine. The final simulation time step shown is where the plasma has reached a quasi-steady state. In the absence of turbulence, one would expect the plasma profile to retain a gradient scale length comparable to the original source (approximated by the first time step profile). However, Fig. 6 shows a relaxation of the gradient scale length over time, not through any modification of the source, but through the nature of the turbulent fluctuations brought on by KH that provide a gradient flattening of the scale length. Here, the experiment is normalized to the final time step of the simulation data for comparisons. The growth rate is about half the size of the drift wave mode with corresponding wavenumbers that are also half those that maximize the drift wave growth rates. Due to its weaker growth rate, we have not definitively observed the sheath mode in the simulations, but it may contribute to structures of order $k_{||} q_s/\sqrt{\nu}$ with corresponding wavenumbers that are also half those that maximize the drift wave growth rates.

IV. LAPD COMPARISONS

Figure 7 shows the total luminosity (left) and the mean-subtracted luminosity (right) looking down the machine from a window at the end of the device using a high-speed CCD camera. Fig. 8 shows a corresponding side by side cut of the total density and the mean-subtracted density.
fluctuations at a given time in the simulations. The scale sizes of the oppositely signed density fluctuations—blobs and holes—in physical units are about 5 cm in both cases and tend to appear to alternate with one another.

Figure 9 also shows CCD measurements compared to GBS simulation data. Here, images from LAPD are taken closer to the area of interest (the cathode edge) using a perisopic mirror arrangement that affords better spatial resolution. To mimic the luminosity line-averaging that is performed by the CCD camera, data from GBS are likewise averaged along $z$ at each time before processing. The comparisons again show similar sized fluctuations at the cathode edge as well as dipole-like structures of over-dense and under-dense fluctuations.

In addition to camera data that capture cross-field profiles at a given instant, time-series data collection in LAPD uses probe measurements taken at various locations within many shots to obtain useful statistical measures of the turbulence. These include the time-averaged level of density fluctuations across the machine radius (Fig. 10), the power-spectrum of these fluctuations (Fig. 11), the skewness or third standardized moment of the distribution (Fig. 14), the 2D cross-field correlation function referenced to a point near the cathode edge (15), and the radial particle flux versus machine radius (Fig. 16). All of these measures, as we now explain, show good agreement with corresponding measurements in the GBS simulations. To assess the role of ion-neutral collisions, we include GBS data both with and without the ion-neutral collision term in Eq. (4).

We use the estimated value of $\nu_{in}/\omega_{ci} = 2 \times 10^{-3}$ for LAPD with a neutral density of $n_n \sim 10^{11} \text{ cm}^{-3}$. Density fluctuations, $\delta n = n - \bar{n}$, are plotted in Fig. 10 as a function of machine radius. The simulations with and
without ion-neutral collisions bracket the LAPD data reasonably well. As one would expect, the largest fluctuations arise near the cathode edge ($r \sim 28$ cm), where the density, temperature, and plasma potential gradients are steepest, and the smallest fluctuation levels occur near the core (small $r$) where the profiles are flattest. The experimental data near the core of the device in this data-set show fluctuations just below 4%. Fluctuations on the order of 1%, which would be in better agreement with the simulations, have also been measured for similar parameters. The addition of ion-neutral collisions causes an absolute amplitude decrease in the fluctuations of about 3%, with the greatest relative decrease near the core. The ion-neutral collisions in the simulation model effectively add a damping term into the vorticity equation which reduces the KH growth rates. The drop in fluctuations both inside and outside of the cathode presumably results from this modest stabilizing effect on KH turbulence.

The power spectrum is calculated by squaring the discrete Fourier transform of $\delta n/n$ in a volume near the cathode edge, $0.22$ m $\leq r \leq 0.28$ m, and using a Hanning window to avoid spectral leakage. The spectrum, as seen from the semilog plot in Fig. 11, shows a linear trend corresponding to an exponential relationship. Here, the focus is on the higher frequencies where the slowly changing equilibrium density is ignored. Exponential frequency power spectra have been associated with coherent structures like holes and blobs in LAPD (Ref. 20). The simulation data in this case are relatively insensitive to ion-neutral collisions and show good agreement with the LAPD measurements.

The total probability distribution function (PDF) probes the tendency for fluctuations to deviate from the mean fluctuation levels. The PDF is approximated by a histogram with unity area and shows the spread of the fluctuations over a given range in the device (highlighted in Fig. 14). Fig. 12 gives the PDF of $\delta n/\text{rms}(\delta n)$ showing an excellent match with the data, having a nearly symmetric Gaussian distribution as fit in Fig. 13. According to the Central Limit Theorem, the mean of any set of variates with any distribution having a finite mean and variance approximates a normal distribution. As expected, the variance of the PDF is close to unity, GBS simulations both with and without ion-neutral collisions appear to have similar PDF shapes. Previous BOUT data, in contrast to LAPD data, favored the mean fluctuations with a peak in the central shape of the PDF.

The third standardized moment of the distribution function is the skewness and in the case of the density is

$$\gamma_1 = \frac{\mu_3}{\sigma^3} = \frac{1}{\left(\frac{1}{2} \sum_n \delta n^3\right)^{3/2}} \left(\frac{1}{\sigma} \sum_n \delta n^2\right)^{1/2}.$$  \hspace{1cm} (10)

This statistical measure is sensitive to the presence of blobs and holes, such as those discussed earlier in the cross-field profiles of the density and temperature. The left-most image of Fig. 8, for example, shows distinct clumps of density outside the central plasma as well as finger-like structures that extend inward to form holes where the density is lower. As a density hole or blob moves past a fixed point, the PDF at that location becomes skewed to the left or right, respectively, reacting to the strong deviation in the density. Skewness data shown in Fig. 14 show such evidence of holes inside the cathode edge and blobs in the profile periphery, in agreement with the LAPD data. These holes and blobs show average sizes of about 5 cm that, as expected in the case of KH vortices, are comparable to the equilibrium density, temperature, and plasma potential scale lengths. The simulations with neutral collisions also show a distinct transition from negative skewness inside the cathode to positive skewness further outward. In agreement with LAPD data, the skewness in the...
range of 0.22 m–0.28 m over which the data for Fig. 12 were collected, is very small. The finding that the Gaussian fit of the PDF in this range is nearly symmetric, is indicative of the smallness of the skewness over which that data were taken.

An issue that calls for further work is the relationship between the plasma profile scale-lengths $L_0$—found here to be comparable to the KH-generated structure sizes—with the LAPD machine parameters, such as $\rho_s$. This relationship is difficult to calculate, since it is determined by the balance between the input from the sources, the cross-field turbulent transport generated by instabilities, such as the KH mode, and the parallel losses along the fieldlines. In Ref. 1, such an analysis, verified by simulations (see Fig. 6(d) of that work), led to the prediction $L_0 \sim 0.1 \sqrt{A \rho_s I_z} \sim 7$ cm, where $L_z = 18$ m is the machine length and $A \sim 3$. On the other hand, in Ref. 19, the alternative estimate $L_0 \sim 10 \rho_s$ was suggested. Given that $\rho_s \sim 1$ cm in the present case, the two estimates lead to roughly the same value. Further work is thus needed to establish the scaling of the profile steepness and blob size with $\rho_s$ and other machine parameters.

Fig. 15 shows the 2D cross-field correlation function of the density fluctuations referenced to a point near the cathode edge and given by

$$P_{xy}(L) = \frac{\sum_{k=0}^{N-1} \sum_{l=0}^{L-1} (x_k-x_l)(y_{k+l}-y_l)}{\sqrt{\sum_{k=0}^{N-1} (x_k-x_l)^2 \sum_{k=0}^{N-1} (y_k-y_l)^2}},$$

(11)

where $P_{xy}$ is the correlation function, $x$ and $y$ are the two arrays to be correlated, and $L$ is the lag between them. In the LAPD, correlation measurements are made between two axially offset probes. The first probe takes data in a plane perpendicular to the magnetic field, with a second multi-tipped probe lying downstream near the end of the device. The lag is chosen so that the correlation between the first probe and the second probe is the greatest. Simulations show correlation lengths of about 5.4 cm which match very well with correlation lengths measured in LAPD. Such cross correlation measurements are consistent with previous LAPD results.

V. CONCLUSIONS

We have carried out global, three dimensional numerical simulations of the large plasma device in the absence of external biasing. The Bohm sheath boundary conditions at the end walls in the simulations lead to the spontaneous formation of plasma potential that is proportional to the electron temperature, $e\phi \sim 3T_e$. This potential is associated with a radially outward electric field and arises physically from the preferential thermal streaming of electrons along the field to the end walls. The radial electric field in turn interacts with the equilibrium axial magnetic field to produce a poloidal $E \times B$ rotation seen in the device under normal operating conditions.

The sheared $E \times B$ rotation of the plasma destabilizes KH modes with $k_\parallel \simeq 0$ that produce turbulence and relatively long-lived, vortical blob-like structures at large radial scales (5–10 cm). Driftwaves and shear modes at smaller spatial scales (5–10 times smaller than the KH structures) are also present but saturate at very small amplitudes compared to the fluctuations arising from the KH activity. The
KH modes, on the other hand, lead to a substantial relaxation of the profiles near the cathode edge, where the profile gradients are typically largest.

The dynamics of the KH vortices produces blobs in the periphery of the device and holes near the core, consistent with LAPD camera data as well as statistical measures, such as the skewness calculated from either probe or simulation data. The density fluctuation power spectrum in the LAPD data and simulations decays exponentially at the same rate—a feature that has been linked to intermittent blob-like structures. Other data-simulation comparisons that show good agreement include the radial dependence and amplitude of the density fluctuations, the PDF of these fluctuations, the spatial correlation lengths near the cathode edge (about 5 cm), and the radial particle flux.

Ion-neutral collisions at theoretically estimated values introduce a damping term in the vorticity equation that, in the case of the time-averaged profiles, is small compared to the shear terms that tend to drive \( \langle J_i \rangle = 0 \) and \( \langle \phi \rangle \sim A(T_e) \). However, this collisional damping does introduce a moderate quieting of the fluctuations in the simulations, particularly near the core, due to its stabilizing effect on the KH modes. The addition of the ion-neutral damping, for example, reduces the radial particle flux by about a factor of two, from values somewhat higher than the LAPD data to values that are somewhat lower. This suppression of the transport leads to a slight steepening in the perpendicular potential profile shown in Fig. 17.

Cases in which the plasma is externally biased may show a stronger deviation in the sheath parameter for collisions and thus impact the plasma rotation through the change in the equilibrium potential profile.

The GBS simulations presented are an improved step towards accurately modeling the LAPD in the low flow regime. LAPD has the capability of applying a bias that can modify the magnitude and sign of the shear flow by an external bias, potentially even eliminating the intrinsic rotation within the plasma. The relative roles of drift waves and KH in these biased configuration runs may differ from those described here and will be the focus of future work.

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**APPENDIX A: VORTICITY SOURCE, \( S_v \)**

The vorticity equation can be obtained from subtracting the electron continuity equation from the ion continuity equation, resulting in a continuity equation for the current. To add a source to the vorticity equation, one assumes a small electron source term representing the primary electronics coming from the hot cathode

\[
\frac{\partial n_i}{\partial t} + \nabla \cdot \left[ n_i (\vec{v}_{E,i} + \vec{v}_{de} + \vec{v}_{pol} + v_i \vec{b}) \right] = S_i, \tag{A1}
\]

\[
\frac{\partial n_e}{\partial t} + \nabla \cdot \left[ n_e (\vec{v}_{E,e} + \vec{v}_{de} + v_e \vec{b}) \right] = S_e + S_{ef}. \tag{A2}
\]

Here, \( \vec{v}_{E,i} \) is the \( E \times B \) drift, \( \vec{v}_{de} \) the ion and electron diamagnetic drifts, \( \vec{v}_{pol} \) the polarization drift, \( v_i \) and \( v_e \) the parallel ion and electron velocities, \( \vec{b} \) the unit vector along the magnetic field, and \( S_{ef} \) represents the source of fast electrons.

Subtracting (A2) from (A1) and assuming quasi-neutrality gives a current continuity equation with a small source term that can be physically thought of as relating to the discharge current each time the plasma is pulsed

\[
\nabla \cdot [n(\vec{v}_{di} - \vec{v}_{de})] + \nabla \cdot (n \vec{v}_{pol}) + \frac{\partial}{\partial z} \left( \frac{J_i}{e} \right) = -S_{ef}. \tag{A3}
\]

Using the Boussinesq approximation and neglecting magnetic curvature terms

\[
\nabla \cdot (n \vec{v}_{pol}) \simeq -\frac{nc}{B_{0}A \eta} \frac{d}{dt} \left( \nabla^2 \phi \right), \tag{A4}
\]

the vorticity equation with \( \omega = \nabla^2 \phi \) can be written as

\[
\frac{d\omega}{dt} = m_e \frac{\omega_{ce}^2}{en} \left[ \nabla \cdot (n(\vec{v}_{di} - \vec{v}_{de}) + \frac{\partial}{\partial z} \left( \frac{J_i}{e} \right) \right] + S_{\omega}. \tag{A5}
\]

**APPENDIX B: CROSS-FIELD ION-NEUTRAL COLLISIONS VIA A PETERSEN CONDUCTIVITY TERM**

The Pedersen conductivity can be written as

\[
\sigma_1 = \sigma_0 \frac{(1 + \kappa) \nu_e^2}{(1 + \kappa)^2 \nu_e^2 + \omega_{ce}^2}, \tag{B1}
\]

where

\[
\sigma_0 = \frac{ne_e^2}{m_e \nu_e}, \tag{B2}
\]

\[
\kappa = \frac{\omega_{ne} \omega_{ci}}{\nu_e \nu_n}, \tag{B3}
\]

\[
\nu_e = \nu_{en} + \nu_{ei}, \tag{B4}
\]
and the collision frequencies for the electrons with neutrals, \( \nu_{en} \), the electrons with ions, \( \nu_{ei} \), and the ions with neutrals, \( \nu_{in} \) are all known from theory or experiment. In the limit where \( \nu_{in}/\omega_i^2 \ll 1 \) (valid for LAPD estimates of \( \nu_{in}/\omega_i \sim 2 \times 10^{-3} \)), the Pedersen conductivity term can be written as

\[
\sigma_1 = \frac{n e^2 \nu_{in}}{m_i \omega_i^2}.
\]  

(5)

Ohm’s law dictates that \( J_\perp = -\sigma_1 \vec{\nabla} \phi \) so that the perpendicular component of current in the current continuity equation becomes

\[
\vec{\nabla} \cdot J_\perp = \vec{\nabla} \cdot (-\sigma_1 \vec{\nabla} \phi),
\]  

(6)

\[
= - \left( \frac{n e^2 \nu_{in}}{m_i \omega_i^2} \right) \omega_c,
\]  

(7)

where it is assumed that \( \sigma_1 \) is not spatially dependent as estimated for LAPD.4

APPENDIX C: MODIFICATION OF BOHM SHEATH FACTOR DUE TO VORTICITY SOURCE

To prevent a buildup of charge in the plasma and maintain quasi-neutrality,

\[
J_{\| \text{cathode}} = J_{\| w_1} + J_{\| w_2},
\]  

(1)

where \( J_{\| \text{cathode}} \) is the discharge current into the source and \( J_{\| w_1 \text{w}_2} \) are the currents out of the near and far walls. Balancing these terms, Eqs. (1) and (2) can be written as

\[
\dot{J} = \frac{J_{\| \text{cathode}}}{q_n \mu_c c_{se}} = 2 - \exp \left( \frac{e}{T_e} (\phi_{se} - \phi_{w1}) \right) \\
\times \left( 1 + \exp \left( \frac{e}{T_e} (\phi_{w2} - \phi_{w1}) \right) \right),
\]  

(2)

where \( n_{se} \) is the plasma density at the sheath edge, \( c_{se} \) is the sound speed at the sheath edge at which ions are assumed to enter, \( \phi_{se} \) is the plasma potential at the sheath edge, and \( \phi_{w1} \) and \( \phi_{w2} \) are the near and far wall potentials. When \( \phi_{w1} = \phi_{w2} \) the exponential factor on the right hand side goes to unity. When \( \phi_{w1} > \phi_{w2} \), the exponential becomes negligible and vanishes. Solving for the plasma potential at the sheath edge, one can show

\[
\phi_{se} - \phi_{w1} = \frac{T_e}{e} \left\{ \ln \left[ \frac{1}{f} (2 - f) \right] \right\},
\]  

(3)

where \( f = 1 \) when \( \phi_{w1} > \phi_{w2} \) and \( f = 2 \) when \( \phi_{w1} = \phi_{w2} \) and \( J < 0 \) since it is modeling an electron beam. Thus, a vorticity source, \( S_v \), which acts as a source of current in the current continuity equation, also effectively shifts the Bohm sheath factor to a lower value when setting the plasma potential.

A modest correction to this calculated potential can likewise be made with the inclusion of ion-neutral collisions which show up in the logarithm of Eq. (C3). Thus, in solving the vorticity equation for a perturbed potential \( \phi = \phi_0 + \phi_1 \), with \( \phi_1 \ll \phi_0 \),

\[
e\phi = \Lambda' T_e,
\]  

(4)

where

\[
\Lambda' = \Lambda - \ln(G)
\]  

(5)

and

\[
G \sim \left( 1 + \frac{S_v}{c_{se}} - \frac{\nu_{in}}{c_{se} \omega_c^2} \phi_0 \right).
\]  

(6)

Note that \( \Lambda' \) is no longer a constant as in the simple sheath case, \( \Lambda = 3 \), but is a function radially dependent upon the temperature, vorticity source, and ion-neutral profile. Near the cathode center, \( \Lambda' \) is approximately 2.2 for simulations both with and without neutrals.

APPENDIX D: SHEATH MODE SCALING AND GROWTH RATE

To eliminate \( k || \ne 0 \) drift waves, the 3D drift-reduced Braginskii equations (Eqs. (3)–(7)) are integrated along the direction of the magnetic field leaving only KH and sheath modes.1 This 2D system of equations can be written as

\[
\frac{dn}{dt} = -\sigma \frac{n C_z}{R} \exp(\Lambda - e\phi/T_e) + S_n, \quad \sigma = \frac{3 R}{2 L_e},
\]  

(1)

\[
\frac{d\nabla^2 \phi}{dt} = \frac{e C_m \Omega_i^2}{e R} \left[ 1 - \exp(\Lambda - e\phi/T_e) \right],
\]  

(2)

\[
\frac{dT_e}{dt} = -\sigma \frac{2 T_e C_z}{3 R^2} \left[ 1.71 \exp(\Lambda - e\phi/T_e) - 0.71 \right] + S_T,
\]  

(3)

where \( L_e \) is the length of the machine along the magnetic field. Here, we focus on the unbiased case where there is no wall potential in the sheath terms, and ion-neutral collisions and sources in the vorticity are neglected.

To find the maximum growth rate of the linear modes found in this 2D system, only the vorticity and temperature equations need to be linearized as they decouple from the continuity equation. Perturbing about an equilibrium quantity, the potential and temperature become

\[
\phi = \phi_0 + \phi_1,
\]  

(4)

\[
T_e = T_{e0} + T_{e1},
\]  

(5)

where \( \phi_1, T_{e1} \ll \phi_0, T_{e0} \). The equilibrium potential and temperature are related through Eq. (9). After expanding Eqs. (2) and (3) in terms of the perturbed quantities, we have

\[
\frac{\partial \nabla^2 \phi_1}{\partial t} + \frac{\nabla H \cdot [\phi_0, \nabla^2 \phi_1]}{B_0} + \frac{\nabla H \cdot [\phi_1, \nabla^2 \phi_0]}{B_0} = \frac{e \mu_0 \Omega_i^2}{e R} \left[ 1 + \frac{T_{e1}}{2 T_{e0}} \right] \left[ 1 - \left( 1 - \frac{e \phi_0}{T_{e0}} \right) \frac{e \phi_0}{T_{e0}} + \frac{e \phi_0}{T_{e0}} \right],
\]  

(6)
where it is assumed that equilibrium terms do not change significantly over the time scales of interest. If we assume our equilibrium quantities are poloidally symmetric and that the local theory is upheld to remove the much smaller radial derivatives of the perturbed quantities, our final linearized equations are

\[
\gamma_k^2 + i \frac{c}{B_0} \phi_i k^3 + \frac{\sigma p}{T_{eq}} \phi_1 - \left[ \frac{\sigma p_0}{T_{eq0}} \right] T_{el} = 0, \\
\gamma + ik_0 \frac{c}{B_0} \phi_0 + 1.71 eD \frac{\phi_0}{T_{eq0}} + 3 \frac{D}{2} T_{el} - ik_0 \frac{c}{B_0} T_{eq0} + 1.71 eD \phi_1 = 0,
\]

where

\[
\gamma = -i \omega, \quad A = \frac{\sigma C_m m \Omega^2}{eR}, \quad D = \frac{2 \sigma C_{00}}{3 R}.
\]

This linear set of equations gives a quadratic solution for the growth rate, \( \gamma \), which is normalized by setting \( \gamma = (C_{00}/R) \gamma_n \) and multiplying through by a factor of \( R^2/\Omega_L^2 \). Putting all of this together, the normalized quadratic equation is

\[
A_n \gamma_n^2 + (B_n + C_n i) \gamma_n + (D_n + E_n i) = 0,
\]

where

\[
A_n = x^2, \\
B_n = \sigma [\Lambda^2 x^4 + 1], \\
C_n = -2 \Lambda \sigma \Lambda x^3, \\
D_n = -\sigma^2 \Lambda^2 x^4 + \sigma^2, \\
E_n = -\sigma \Lambda \Lambda \Lambda x^3,
\]

with

\[
x = k_y \rho_s, \quad \sigma = R/L_\phi, \quad \Lambda' = \left( \frac{3.42}{3} \Lambda + 1 \right).
\]

Our interest is in the unstable modes found from the real-part of the growth rate. In the solution for \( \gamma_n \), we drop the imaginary terms and use De Moivre’s theorem with an appropriate Argand diagram to find the real-part of the positive-definite quadrature term in the quadratic formula. The real-part of the growth rate is then found to be

\[
\text{Re}(\gamma_n) = -\frac{4.42 \sigma}{2} \frac{1}{2 \sqrt{2} x^2 + \sqrt{x^2 + \beta^2}}, \quad (D16)
\]

where \( \beta = 12 \sigma \Lambda x^3 \).

An even simpler form is arrived at by dropping the \( \sigma^2 \) term under the quadrature in Eq. (D16), which is of order magnitude smaller than the rest. A general form of the sheath mode scaling then becomes

\[
\text{Re}(\gamma_n)_{\text{max}} = \frac{3 R}{4 L_z} \left[ \frac{\Lambda L_z}{12 L_\phi} \right]^{2/3} - \Lambda'.
\]

For simulated parameters, the normalized real-part of the linear growth rate has a maximum of \( \gamma \sim 0.635 \) when \( x = k_y \rho_s \sim 0.268 \) as shown in Fig. 18.

To find a tractable analytical solution for \( x \) that maximizes the growth rate, each term and sub-term in Eq. (D16) are compared at the numerically calculated peak growth rate and terms that are two orders of magnitude smaller than the rest are dropped. Doing this gives us a tri-quadratic equation in \( x \)

\[
\text{Re}(\gamma_n) = -\frac{4.42 \sigma}{2} + \frac{1}{2 \sqrt{2} x^2 + \sqrt{x^2 + \beta^2}} + 12 \sigma \Lambda x^3,
\]

which gives a maximum growth rate for the sheath mode

\[
\text{Re}(\gamma_n)_{\text{max}} = \frac{3 R}{4 L_z} \left[ \frac{\Lambda L_z}{12 L_\phi} \right]^{2/3} - \Lambda'\right).
\]

An even simpler form is arrived at by dropping the \( \sigma^2 \) term under the quadrature in Eq. (D16), which is of order magnitude smaller than the rest. A general form of the sheath mode scaling then becomes

\[
\text{Re}(\gamma_n)_{\text{max}} = \frac{3 R}{4 L_z} \left[ \frac{\Lambda L_z}{12 L_\phi} \right]^{2/3} - \Lambda'.
\]