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ABSTRACT
A model for the steady-state operation of an emissive cathode is presented. The cathode, biased negative with respect to a cold anode, emits electrons thermionically and is embedded within a large magnetized-plasma column. The model provides formulas for the spatial shape of the global current system, the partition of potential across the plasma–sheath system, and the effective plasma resistance. The formation of a virtual cathode is explored, and an analytical expression for the critical operating conditions is derived. The model is further developed to include the self-consistent increase in plasma temperature which results from thermionic injection. In a companion paper [S. Jin et al., Phys. Plasmas 26, 022105 (2019)], results from transport experiments in the Large Plasma Device at the University of California Los Angeles are compared with this model, and excellent quantitative agreement is achieved.

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I. INTRODUCTION

An issue of importance in the study and confinement of magnetized-plasmas is how to control and adjust internal features of the plasma via the implementation of external boundary conditions. This topic has particular relevance for shear-stabilization of turbulence \(^{(1,2)}\) and the use of active-control in plasma confinement. The primary difficulty in addressing this question is that the internal features of a plasma are separated from the external environment by a small, collisionless region where nonlinear physics dominates. The sheath, as it is termed, is the barrier through which all external input must pass before reaching the bulk of the plasma. This makes sheath-physics intrinsically tied to the implementation of any boundary-critical scheme.

Of specific interest is the boundary of a heated emissive cathode, biased negative with respect to a cold anode. On the surface of the heated cathode, bound electrons with thermal energy large enough to overcome the work function of the material “boil” off of the surface, producing a net flow of electric charge. The resulting emission is termed thermionic, and its behavior as a function of the material’s surface temperature and work function \(^{(3)}\) was first identified by Richardson in 1901, which later led to his receiving of the 1928 Nobel Prize in Physics.

In vacuum, the space-charge density carried by the thermionic electrons creates its own electric field, which then acts back on the electrons. The self-consistent solution was first worked out by Child in 1911 \(^{(4)}\) and was later extended by Langmuir in 1913; the resulting Child-Langmuir law \(^{(5,6)}\) dictates that current injected into vacuum is space-charge limited, implying the existence of an upper-bound for the total thermionic current that depends only on the geometry of the system and the 3/2 power of the bias voltage.

The space-charge limit refers to the special situation of injection into a vacuum; however, dramatic modifications to the limit occur when considering injection into a quasi-neutral plasma. In this situation, the ion-current from the plasma causes partial cancellation of the space-charge, which can increase the upper-bound for the total current by several orders of magnitude. The charge injected into the plasma seeks quasi-neutrality and forms a global current system, accompanied by a self-consistent electric field.

The topic of thermionic cathodes has been the focus of several dedicated experimental, theoretical, and numerical research efforts. \(^{(7-16)}\) Analytical studies by Pekker and Hussary \(^{(10,11)}\) derived a set of self-consistent two-fluid boundary conditions for the sheath region between a thermionic-emitting cathode and a neutral plasma. The derived boundary conditions were further implemented to explore a zero-dimensional model of the cathode spot and the formation of a virtual cathode.

Numerical studies by Campanell and Umansky \(^{(13,14)}\) demonstrated using a continuum kinetic solver the occurrence of
several mode transitions associated with the dynamical formation and destruction of a virtual cathode by charge exchange processes within the sheath.

Studies of the short argon arc performed by Khrabry et al.\textsuperscript{15,16} developed a self-consistent analytical model composed of sub-models for the near-electrode regions and arc column, including the effect of heat transfer in the cylindrical electrodes. The derived model demonstrated good agreement with both simulation and experiment.

The model considered here matches an analytical solution\textsuperscript{17} of the global, resistive plasma potential to the boundary\textsuperscript{18} at the sheath-plasma interface, allowing self-consistency within a set of defined constraints. The bulk of the plasma is assumed to be larger than the mean-free-path of the charge carriers, so that, in the steady-state, the plasma is neutral, and a well-defined conductivity tensor exists at each point. The sheath region, assumed smaller than the mean-free-path, is dominated by collisionless physics involving the injected thermionic electrons and the plasma-ions and is approximated as one-dimensional.

Detailed results from transport experiments\textsuperscript{15} conducted in the Large Plasma Device (LAPD)\textsuperscript{19} at the University of California, Los Angeles (UCLA) and comparisons to the model are presented in a companion paper.\textsuperscript{16} It is found that the model exhibits excellent quantitative agreement with the electron physics in the experiments; however, there are subtleties in the field-aligned ion-flow which remain elusive.

This manuscript is organized as follows: Section II details the model-equations for the plasma potential and the associated boundary conditions. Section III states the analytical solution to the problem outlined in Sec. II for a disk-shaped cathode. Section IV derives model-constraints and extends the model to account for virtual cathode formation. Section V offers an extension of the model for a ring-shaped cathode, which corresponds to the configuration used in Ref. 20. Section VI further expands the model to include the self-consistent calculation of the increase in plasma-temperature. Conclusions are given in Sec. VII.

II. PLASMA POTENTIAL MODEL

The geometry under consideration models transport experiments conducted in the LAPD: the z-axis points along the confining magnetic field with the cathode located at \( z = 0 \) and the anode at \( z = L \). The radial extent of the anode is assumed to be much larger than the cathode, so effects associated with the finite radial size of the cathode may be neglected.

In the steady-state, the perpendicular and parallel currents in the bulk of the plasma are expressed in terms of the plasma potential \( \phi \) via Ohm’s law

\[
\mathbf{j}_\perp = -\sigma_\perp \nabla_\perp \phi, \quad j_\parallel = -\sigma_\parallel \nabla_\parallel \phi. \tag{1}
\]

The continuity of charge and quasi-neutrality yield an equation for the plasma potential

\[
\sigma_\perp \nabla_\perp^2 \phi + \sigma_\parallel \nabla_\parallel^2 \phi = 0, \tag{2}
\]

where the electrical conductivities are assumed to be spatially uniform within the plasma.

For a partially ionized and strongly magnetized plasma, such as the afterglow of an LAPD discharge, the dominant contribution to the perpendicular conductivity \( \sigma_\perp \) is due to collisions between ions and neutrals

\[
\sigma_\perp = 2.13 \frac{e^2 n_{\text{ion}}}{M_0 \tau}, \tag{3}
\]

and the parallel conductivity \( \sigma_\parallel \) results from Coulomb collisions between electrons and ions

\[
\sigma_\parallel = 1.96 \frac{e^2 \tau_e}{m}. \tag{4}
\]

Here, \( e \) is the unit of charge, \( M \) and \( m \) are the respective ion and electron masses, \( n \) is the plasma density, \( \Omega_e \) is the ion-cyclotron frequency, \( \tau_{\text{ion}} \) is the ion-neutral collision frequency, and \( \tau_e \) is Braginskii’s electron collision time.

The assumption of uniform parallel electrical conductivity in the bulk of the plasma treats spatial gradients in electron temperature as negligible, simplifying the problem and allowing an exact analytic treatment. For the ring cathode experiment,\textsuperscript{16,20} the variation of plasma potential between the cathode and the anode is a factor of 20 greater than the variation of electron temperature, supporting the use of this assumption for those experimental conditions.

In the afterglow of an LAPD discharge,\textsuperscript{21} the plasma potential and associated current system are established on a timescale of less than 200 \( \mu \)s. The plasma density, however, decays due to axial outflow on a timescale greater than 5 ms. This separation of temporal scales allows the plasma density to be treated as approximately constant for the time scale on which the plasma potential establishes a quasi-steady state, as described by Eq. (2).

A. Boundary conditions

The surfaces of the cathode and anode are separated from the bulk of the plasma by small sheath regions with sizes on the order of the Debye length. An external bias voltage, \( V_b \), applied between the two surfaces is partitioned into three regions: the cathode sheath, the bulk of the plasma, and the anode sheath. The respective potential drops, \( \phi_{\text{cath}}, V_p, \) and \( \phi_{\text{anode}} \), satisfy

\[
V_b = \phi_{\text{cath}} + V_p - \phi_{\text{anode}}, \tag{5}
\]

where all four quantities are taken to be positive.

The total current injected into the plasma from a disk-shaped thermionic cathode of radius \( a \)

\[
I_{\text{tot}} = 2\pi \int_0^a r \, dr \left[ \sigma_\parallel \nabla_\parallel \phi \right]_{z=0}, \tag{6}
\]

is matched to the one-dimensional current density in the cathode sheath, providing a self-consistent boundary condition at the emissive cathode

\[
I_{\text{tot}} = I_1 (1 - e^{-\frac{\Gamma}{2}} \phi_{\text{cath}}) + I_{\text{eth}}, \tag{7}
\]

where

\[
\Gamma \equiv -\ln \sqrt{\frac{2\pi m_e}{m_i}} \approx 3.5, \tag{8}
\]
which is a standard sheath parameter that yields the floating potential in the absence of thermionic emission. The thermionic current density \( j_{\text{th}} \) is assumed to obey Richardson’s law

\[
 j_{\text{th}} = A_e \cdot C_e T_s^2 \exp \left( -\frac{\phi_{\text{out}}}{T_s} \right),
\]

where \( A_e \) is the surface area of the cathode, \( T_s \) is the surface temperature, \( \phi_{\text{out}} \) is the work function of the material, and \( C_e \) is the material-dependent Richardson constant. The ion saturation current, \( j_i = A_e \cdot e n_i \), is set by the density and ion sound speed, \( c_s = \sqrt{T_e/M} \), at the sheath–plasma interface.

As discussed in Sec. IV, Eq. (9) neglects virtual cathode formation. For large thermionic currents, the self-consistent electric field produced by the thermionic electrons limits this value. Section III assumes that the conditions are such that no self-limiting in the cathode sheath occurs.

For a large, cold (i.e., non-emissive) anode, the current density in the anode sheath is taken to be small, allowing the potential drop to be approximated by \( \phi_{\text{anode}} = T_e \Gamma/e \). At the interface between the anode sheath and plasma, the boundary condition is taken to be

\[
 \phi|_{z=L} = 0,
\]

which sets the boundary condition at the interface between the cathode sheath and plasma

\[
 \phi|_{z=0} = -V_p, \quad r < a.
\]

### III. THE STEADY-STATE PLASMA POTENTIAL OF AN EMISSIVE DISK

The solution to Eq. (2) can be written as an integral in Fourier–space

\[
 \phi(r, z) = \int_0^\infty dk \tilde{j}_0(kr) \tilde{\phi}(k, z),
\]

where \( \tilde{j}_0 \) is the zeroth-order Bessel function. The function \( \tilde{\phi} \), the Hankel transform of \( \phi \), is chosen to have a form that ensures that boundary conditions (10) and (11) are enforced

\[
 \tilde{\phi}(k, z) = -\frac{2V_p a}{\pi k} j_0(ka) e^{-k(z)|z|} e^{-k(z)2L} 1 - e^{-k(z)2L},
\]

where \( k_0(z) \equiv k \sqrt{\sigma_r/\sigma_\parallel} \) and \( J_0 \) is the zeroth-order spherical Bessel function.

Expanding the geometric series in the denominator and completing the \( k \)-space integration allow the solution to be expressed as the rapidly converging series

\[
 \phi(r, z) = \frac{2V_p}{\pi} \sum_{n=-\infty}^{\infty} \left( \sin^{-1} \frac{a}{x_{n+}} - \sin^{-1} \frac{a}{x_{n-}} \right),
\]

where the geometric quantities are defined in cylindrical coordinates as

\[
 x_{n\pm} = \frac{1}{2} \left( \sqrt{b_{n\pm}^2 + (r + a)^2} + \sqrt{b_{n\pm}^2 + (r - a)^2} \right),
\]

and

\[
 b_{n\pm} = \sqrt{\frac{a}{\sigma_r} (2nL + |z - L\pm L|)}.
\]

The value of \( V_p \) is not externally set but must be self-consistently calculated from the externally applied bias \( V_s \). This is accomplished by matching the resistive current in the plasma to the current in the one-dimensional cathode sheath. Using Eq. (14) to evaluate the total current in Eq. (6) gives (see Appendix A)

\[
 I_{\text{tot}} = \frac{1}{\tilde{g}(s)} \frac{\pi a^2 \sigma_r}{L} V_p.
\]

The dimensionless quantity \( s \) specifies the global shape of the plasma potential

\[
 s = \frac{a}{L} \sqrt{\frac{\sigma_r}{\sigma_\parallel}},
\]

where small \( s \) corresponds to a potential that exponentially decays before reaching the anode, and large \( s \) corresponds to a potential that is linear between the cathode and the anode.

The function

\[
 \tilde{g}(s) = \frac{s}{4} \left[ \sum_{n=-\infty}^{\infty} \left( \frac{2}{\sqrt{1 + \frac{1}{1 + s^2}} \frac{1}{s^2}} \right)^{-1} \right]
\]

is displayed in Fig. 1. It scales the current in accordance with the geometry of the potential, as set by \( s \). In the limit of small \( s \), \( \tilde{g}(s) \) linearly increases with its argument, while in the opposite limit of large \( s \), it saturates to unity.

Rearranging Eq. (17) into the form

\[
 g(s) = \frac{\pi s}{4} \sum_{n=-\infty}^{\infty} \left( \frac{2}{\sqrt{1 + \frac{1}{1 + s^2}} \frac{1}{s^2}} \right)^{-1}
\]

![Fig. 1. Plot of Eq. (19).](image)
\[ V_p = I_{\text{tot}} R_p \]  

(20)

motivates the definition of the plasma resistance

\[ R_p \equiv g(s) \frac{L}{\pi u^2 a_j}. \]  

(21)

Using Eqs. (5) and (20), \( \phi_{\text{cath}} \) is written in terms of \( I_{\text{tot}} \)

\[ \phi_{\text{cath}} = \frac{T_e}{e} \Gamma + V_b - I_{\text{tot}} R_p \]  

(22)

and eliminated from Eq. (7), yielding the transcendental equation

\[ I_{\text{tot}} = I_1(1 - e^{\psi_0 R_p} - \psi_0) + I_{\text{eth}}. \]  

(23)

The solution to Eq. (23) (see Appendix B) expresses the total current and potential drop across the plasma in terms of the externally applied bias

\[ \psi_c = \psi_c - \ln J_i, \quad \psi_c^0 = \psi_c - \ln J_i, \]  

(26)

and rewriting Eqs. (22) and (24), enables the potential drop across the cathode sheath to be expressed in the form

\[ \psi_c = \Gamma - (p - \psi_c^0) + W(e^{\psi_c - \psi_c^0}), \]  

(27)

where the parameter

\[ p = J_i + J_{\text{eth}} \]  

(28)

is the scaled, maximum current, corresponding to the global maximum of Eq. (24), for positive \( J_i \).

The behavior of Eq. (27) is shown in Fig. 3, where the potential drop across the cathode sheath \( \psi_c \) is displayed with respect to the argument \( p - \psi_c^0 \). In the case of large applied bias with respect to the parameter \( p (\psi_c^0 \gg p) \), most of the applied bias appears across the cathode sheath. This is termed the primary regime since the source of the temperature increase in the plasma is the ballistic scattering of electrons accelerated in the sheath potential (i.e., primary heating). In the opposite limit

\[ \begin{array}{c}
\text{Primary regime} \\
\text{Ohmic regime}
\end{array} \]

\[ \begin{array}{c}
\text{increasing applied bias} \\
\text{increasing thermionic current}
\end{array} \]

FIG. 2. Radial and axial (r, z) contour of potential given by Eq. (14) and the associated current-system for a disk-shaped cathode configuration. Lines of current begin on the anode and terminate on the cathode. The situation corresponds to a plasma with \( s = 1/3 \). Arrows depict the local current density vector found from the evaluation of the derivatives in Eq. (1). Rotating about the line \( r = 0 \) gives the three-dimensional potential and current-system. The dashed black line at \( z = L \) corresponds to the interface between the plasma and the anode sheath, where the potential vanishes, and the solid red line at \( z = 0 \) corresponds to the interface between the cathode sheath and plasma, where the potential has the value \(-V_p\).

A. Partitioning of potential

Defining the quantities

\[ \psi_c^0 = \psi_c - \ln J_i, \quad \psi_b^0 = \psi_b - \ln J_i, \]  

(26)

and rewriting Eqs. (22) and (24), enables the potential drop across the cathode sheath to be expressed in the form

\[ \psi_c = \Gamma - (p - \psi_c^0) + W(e^{\psi_c - \psi_c^0}), \]  

(27)

where the parameter

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\text{Ohmic regime}
\end{array} \]

\[ \begin{array}{c}
\text{increasing applied bias} \\
\text{increasing thermionic current}
\end{array} \]
the quantity \( p \), termed the Ohmic regime, the applied bias appears across the plasma and the main source of temperature increase is due to Ohmic–heating in the plasma. For a given experimental configuration, a simple comparison of the applied bias \( \psi'_b \) with the quantity \( p \) allows for straightforward determination of the operational regime.

As \( p \) is determined by the resistive geometry, the thermionic–electron current, and the ion saturation current, changes in the plasma parameters over the time-duration of an experiment may cause changes in the operational regime. The critical condition where this occurs is \( p = \psi'_b \).

In terms of \( p \) and \( \psi'_b \), the scaled total current has the form

\[
J_{tot} = p - W (e^{p - \psi'_b}) .
\]  

(29)

Figure 4 shows the scaled total current given by Eq. (29) as a function of the scaled, maximum current \( p \), and applied bias \( \psi'_b \). As is evident from the figure, the total current saturates in both \( p \) and \( \psi'_b \). For fixed \( \psi'_b \), increasing the value of \( p \) allows more of the applied bias to appear across the plasma, until the point where all of the bias is across the plasma; a further increase in \( p \) does not increase the potential drop across the plasma nor the total current. For fixed \( p \), applied bias will appear across the plasma until \( \psi'_b = p \), at which point, additional applied bias will appear across the cathode sheath. The left-half of the pyramid corresponds to the primary regime and the right-half to the Ohmic.

B. Asymptotic limits: Resistance

In the limit \( s \gg 1 \) (i.e., \( \sqrt{\sigma_{||}/\sigma_\perp} \gg L/a \)), Eq. (21) gives

\[
R_p \approx \frac{L}{\pi a^2 \sigma_{||}} .
\]  

(30)

which corresponds to small cathode–anode spacing or large ratio of parallel to perpendicular conductivity; its form is equivalent to the resistance of a wire with conductivity \( \sigma_{||} \).

In the opposite limit \( s \ll 1 \) (i.e., \( \sqrt{\sigma_{||}/\sigma_\perp} \ll L/a \)), the expression

\[
R_p \approx \frac{1}{4a \sqrt{\sigma_{||}/\sigma_\perp}}
\]  

(31)

corresponds to large cathode–anode spacing.

For experimental conditions in the afterglow of an LAPD discharge, the quantity \( \sqrt{\sigma_{||}/\sigma_\perp} \) has a numerical value of about 200. Assuming that the cathode–anode spacing spans the length of the machine (i.e., \( L \approx 15 \) m) and the cathode radius is small (i.e., \( a \approx 5 \) cm), the ratio \( L/a \) is 300, which places afterglow experiments in an intermediate regime.

Figure 5 displays the plasma resistance as a function of temperature in log-log format for four different ratios of \( L/a \). In the limit of large temperature, Eq. (30) indicates that the resistance scales as \( T_e^{-3/4} \), and in the limit of small temperature, it scales as \( T_e^{-3/2} \). The parameters are chosen to match afterglow conditions in the LAPD with \( n = 4 \times 10^{19} \) cm\(^{-3} \), and neutral density on the order of \( n_0 \approx 4 \times 10^{12} \) cm\(^{-3} \).

C. Asymptotic limits: Current

Assuming \( J_{tot} > 0 \), the argument of the Lambert–W function is restricted to the interval

\[
0 < e^{p - \psi'_b} < p e^p .
\]  

(32)

FIG. 4. Operational parameter space of total current in the absence of a virtual cathode, as given by Eq. (29). The left-half of the pyramid corresponds to the primary regime and the right-half to the Ohmic.

FIG. 5. Log-log display of plasma resistance as a function of temperature for different geometries. At low temperatures, the plasma resistance scales as \( T_e^{-3/4} \), while at higher temperatures, it scales as \( T_e^{-3/2} \). The parameters are chosen to match LAPD afterglow conditions.
The lower-end of this interval corresponds to a large applied bias (i.e., \( \psi_b' > p \)). Expanding the Lambert-W function in Eq. (24) for small argument gives

\[
J_{\text{tot}} \approx p - e^{p - \psi_b'}.
\]

which has the same form as the current to a Langmuir probe. Saturation occurs in the situation that the exponential can be neglected; in this case, the scaled total current and potential drop across the plasma are simply equal to \( p \). Any additional externally applied bias in this regime appears across the cathode sheath.

The upper-end of the interval in Eq. (32) corresponds to an applied bias that is only slightly larger than \( \ln \frac{1}{b} \). Expanding \( J_{\text{tot}} \) about \( \psi_b' + \ln p \) gives

\[
J_{\text{tot}} \approx \frac{p}{1 + b} (\psi_b' + \ln p).
\]

This implies a linear relationship between the current and applied bias, which is indicative of a resistive potential drop.

Thus, for a small applied bias, \( \psi_b' < p \), the system acts like a resistor, while for a large applied bias, \( \psi_b' > p \), the system acts like a Langmuir-probe. The expression given by Eq. (29) provides the continuous connection between these two regimes.

**IV. VIRTUAL CATHODE FORMATION**

The self-limiting of thermionic current is the central concept underlying the well-known Child–Langmuir law, which derives the existence of an upper-bound for current injected into vacuum, dependent on the applied voltage to the 3/2-power. For injection into a neutral plasma, an analogous self-limiting procedure occurs due to the interplay between the plasma-current and the thermionic electron current.

For large thermionic current and small ion saturation current, the space-charge of the ions is insufficient to cancel the space-charge of the thermionic electrons, and the result is the formation of a virtual cathode, a minimum in the sheath-potential, which limits the total thermionic current. The critical value of the thermionic current for virtual cathode formation can be determined by solving the one dimensional Poisson’s equation in the cathode sheath under the constraint of monotonicity (see Appendix C). The scaled critical value of thermionic current, \( J_{\text{crit}} \), as a function of the potential drop across the cathode sheath, \( \psi_c \), takes the form

\[
J_{\text{crit}} = J_1 \sqrt{\frac{m e^{-\psi_c}}{\psi_c} + \sqrt{1 + \frac{2 \psi_c}{\psi_c} - 2} \frac{1}{\sqrt{2 \psi_c}}}. \quad (35)
\]

Replacing the thermionic current with its critical value, and using Eqs. (3) and (20) to eliminate \( I_{\text{tot}} \) from Eq. (7), gives the transcendental equation for \( \psi_c \)

\[
\psi_b = J_1 \left[ 1 - e^{\Gamma - \psi_c} + \sqrt{\frac{m e^{-\psi_c}}{\psi_c} + \sqrt{1 + \frac{2 \psi_c}{\psi_c} - 2} \frac{1}{\sqrt{2 \psi_c}}} \right] + \psi_c - \Gamma. \quad (36)
\]

In the case that \( J_{\text{crit}} > J_{\text{eth}} \), a virtual cathode forms, and Eq. (36) must be numerically solved for \( \psi_c \) given values of \( \psi_b \) and \( J_1 \). With the numerically computed value of \( \psi_c \), the total current is found using Eq. (22)

\[
J_{\text{tot}} = \Gamma + \psi_b - \psi_c. \quad (37)
\]

To illustrate the behavior of the virtual cathode, it is helpful to consider the limit where the thermionic current is infinite. In this case, the injected current is always limited by the potential drop across the cathode sheath, as given by Eq. (35), and the resulting total current depends only on the applied bias and ion saturation current.

The parameter dependence of total current for the case of infinite thermionic–current is shown in Fig. 6. The vertical values are generated by numerically solving Eq. (36) for varying \( \psi_b \) and \( J_1 \). For small values of the ion saturation current \( J_i \), there is insufficient ion space-charge to cancel the space-charge of the thermionic electrons, and so the total current is limited. For large ion saturation current, the system behaves like a resistor, with the total current linearly dependent on the applied-bias.

Figure 7 shows the effect of virtual cathode formation on the total current for three values of the ion saturation current. Panels (a), (b), and (c) show the effect on the total current for \( J_i \) values of 3, 1, and 0.5, respectively. The dotted black line indicates the critical threshold for virtual cathode formation; points in the parameter space that lie to the right of the line correspond to virtual cathode formation and points to the left of the line have monotonicity in the sheath. For values of \( J_i \) greater than unity, the pyramid–shape is essentially unaffected by the space-charge cutoff; however, for \( J_i \) less than unity, the pyramid can be significantly altered.

For the ring-cathode geometry discussed in Sec. V, the value of \( J_i \) is estimated to be greater than 3, indicating that the formation of a virtual cathode plays only a minor role in current-reduction. However, as the density decays due to outflow at the machine ends, the value of \( J_i \) decreases, and the
virtual cathode may play an important role. The point where virtual cathode formation begins to significantly deviate from monotonic behavior can be estimated by setting $J_i = 1$.

It should be noted that the type of virtual cathode considered here may not always be applicable. In particular, trapped ions created by charge exchange processes within the sheath region tend to smooth-out the minimum in the sheath potential, eventually leading to the formation of an inverse sheath mode. For experiments in the afterglow of the LAPD, the mean-free-path of charge carriers is estimated to be much larger than the size of the sheath region, and so, for the present considerations, the dynamic features and mode transitions associated with charge exchange in the sheath are neglected.

### A. Threshold conditions

In the design and operation of emissive cathodes for laboratory experiments, the conditions that lead to virtual cathode formation are important to consider. Of specific interest is the minimum external-bias $\psi_b^*$ needed to prevent the formation of a virtual cathode for a given thermionic current, or conversely, the maximum thermionic current that can be drawn from an emissive cathode, given an applied external-bias.

The external-bias voltage corresponding to criticality is found by numerically calculating the critical value of the cathode potential drop $\psi_c^*$ for a given $J_{eth}$ in Eq. (35); then, the numerically computed value of $\psi_c^*$ is used to evaluate Eq. (36), yielding the critical value $\psi_b^*$.

An analytic approximation may be obtained by solving the asymptotic limits of Eq. (35) and patching the resulting solutions. The limit of small $\psi_c^*$ gives

$$\psi_{c1}^* = \left(3\sqrt{2}\right)^{1/5},$$

which can be iterated to give the correction

$$\delta\psi_{c2} = \frac{1}{J} \frac{e^{-\psi_{c2}}}{\sqrt{2\psi_{c2} + \sqrt{1 + 2\psi_{c2}}} + e^{\psi_{c2}}}.$$  

Patching Eq. (38) with the sum of Eqs. (39) and (40) produces the approximation for $\psi_b^*$

$$\psi_b^* = e^{-\triangle} \psi_{c1}^* + (1 - e^{-\triangle})(\psi_{c2}^* + \delta\psi_{c2}).$$

where the patching parameter $\triangle \approx 20$.

Evaluating $\psi_b^*$ in Eq. (39) using Eq. (41) yields an accurate approximation for $\psi_b^*$ without the need for numerical root-finding

$$\psi_b^* = \psi_c^* + J \left[1 + J \sqrt{\frac{M}{m}} - e^{-\psi_c^*}\right].$$

Thus, at a given thermionic current $J$, an applied bias satisfying $\psi_b^*_b < \psi_b^*$ results in the formation of a virtual cathode, which, in turn, limits the total thermionic current to the critical value.

Figure 8 shows a comparison of the numerical solution (blue line) to the analytical approximation (dotted red line) for the critical values of $\psi_b^*$ and $\psi_c^*$. Each point on the $J$-$\psi_b^*$-plane corresponds to a different operating condition. Points above the threshold line correspond to situations with monotonic sheaths; points below the line correspond to situations where a virtual cathode is formed.

The total current as a function of the cathode-surface temperature for fixed externally applied bias is displayed in Fig. 9. Increasing the surface temperature of the cathode increases the thermionic current in accordance with Richardson’s Law given by Eq. (9). Below the critical surface temperature $T_{surf} \approx 1625\text{C}$,
no virtual cathode exists and the current obeys Eq. (29). Above the critical surface temperature, a virtual cathode forms and the thermionic current is limited to its critical value. Further increase in $T_c$ beyond the critical temperature does not result in additional current. The black diamonds correspond to data collected in LAPD transport experiments with the ring-shaped geometry discussed in Sec. V.2.0.

A point worth mentioning is that the density is not considered to be a variable as the bias or emission is swept. For situations where the injected electrons significantly ionize neutrals in the plasma, the self-consistent increase in density due to ionization can lead to currents exceeding the threshold conditions discussed here.

### V. EXTENSION TO A RING-SHAPED CATHODE

Recent transport experiments conducted in the LAPD$^{18,20,21}$ have a novel geometry consisting of an annular-shaped thermionic cathode, biased negative with respect to a cold, distant anode. The cathode is constructed by attaching a carbon mask of radius $r = 2$ cm to a disk-shaped cathode of radius $r = 3$ cm. The inner-mask prevents the emission of thermionic electrons, which leaves an annular region of emission. Applying a bias with respect to an anode results in a current system that peaks off-axis, taking the shape of a hollow cylinder.

This section extends the model previously considered to the geometry of the ring-shaped cathode. This is accomplished by superimposing an additional solution with the one considered in Sec. III. The important results for the disk-geometry carry over to the ring-geometry with the simple replacement of the plasma resistance.

The general solution to Eq. (2) satisfying the boundary condition at the anode can be expressed as

$$\phi(r, z) = \int_0^\infty \text{d}k A(k) J_0(kr) \frac{e^{ikb|z|} - e^{ik(|z| - 2L)}}{1 - e^{ikb|z|}}. \quad (43)$$

For a disk-shaped cathode of radius $b$, the function $A(k)$ has the form

$$A_{\text{disk}}(k) = -\frac{2V_p}{\pi} b J_0(kb), \quad (44)$$

while for a circular hole of radius $a$ in a conducting plane

$$A_{\text{hole}}(k) = -\frac{2V_p}{\pi} a j_1(ka). \quad (45)$$

The superposition of the above two solutions gives an approximation for the potential formed by a conducting annulus of inner radius $a$ and outer radius $b$

$$A_{\text{ring}}(k) = -\frac{2V_p}{\pi b^2} (b^2 j_0(kb) - a^2 j_1(ka)). \quad (46)$$

Expanding the geometric series in the denominator of Eq. (43) and completing the $k$-space integration give

$$\phi_{\text{ring}}(r, z) = \sum_{n=0}^\infty \left[ \phi_{n-1}^{\text{disk}} - \phi_{n+1}^{\text{disk}} - \frac{a^2}{b^2} (\phi_{n-1}^{\text{hole}} - \phi_{n+1}^{\text{hole}}) \right]. \quad (47)$$

where

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**FIG. 8.** Comparison of numerical solution (solid blue line) to analytical approximation (dotted red line), for the critical values of $\psi_0$, panel (a), and $\psi_c$, panel (b), for a value of $J = 3$. Points on the $J-\psi$ plane which are above the line correspond to situations where the sheath is monotonic; points below the line correspond to where a virtual cathode exists.

**FIG. 9.** Dependence of total cathode-current on the cathode-surface temperature. The solid blue line is the expression given in Eq. (24), dashed red and green lines correspond to the small and large argument expansions of the $W$ function, respectively, and the dashed black line indicates the formation of a virtual cathode. Temperature is in units of Celsius, and current is in units of Amps. The black diamonds correspond to the measured LaB$_6$ current values in the ring-cathode experiment (red points in Fig. 3 of the companion paper$^{20}$).
The subscripted geometric quantities are

\[ x_{\text{in, n.z.}} = \frac{1}{2} \left( \sqrt{b_{n,z}^2 + (r + \xi)^2} + \sqrt{b_{n,z}^2 + (r - \xi)^2} \right), \]

where the subscript \( \xi \) denotes either the inner radius \( a \) or outer radius \( b \), and \( b_{n,z} \) has the same form as in Sec. III.

Figure 10 shows the contour plot of a particular potential given by Eq. (47). The case shown is chosen to match the parameters for the ring-cathode experiments in LAPD. Arrows correspond to the local current density vector, tangent to the flow lines of current depicted as dashed black lines. The solid colored curves correspond to equipotentials scaled to \( V_p/10 \). The horizontal solid red line at \( z = 0 \) shows the cathode location, and the horizontal black dashed line at \( z = L \) is the anode.

The total current injected into the plasma from a ring-shaped cathode

\[ I_{\text{tot}} = 2\pi \int_a^b r \, dr \, \left[ \sigma_r \nabla \phi \right]_{z=0}, \]

corresponds to Eq. (6) for the disk-shaped case.

Using Eq. (47) to evaluate to total current in Eq. (51) gives

\[ I_{\text{tot}} = \frac{1}{g(s_a, s_b)} \frac{\pi (b^2 - a^2)}{L} V_p, \]

The quantities \( s_a \) and \( s_b \) determine the global shape of the plasma potential and are defined by

\[ s_a = \frac{a}{b} \sqrt{\frac{\sigma_{\parallel}}{\sigma_{\perp}}}, \quad s_b = \frac{b}{L} \sqrt{\frac{\sigma_{\parallel}}{\sigma_{\perp}}}, \]

The function \( g(s_a, s_b) \) has the form

\[ g(s_a, s_b) = \frac{\pi (s_a^2 - s_b^2)}{4s_b} \left[ 1 + 2 \sum_{n=1}^\infty K_n \right]^{-1}, \]

where

\[ K_n = \frac{\sqrt{2}}{\sqrt{u_n + \sqrt{u_n^2 + s_b^2/n^2}}} - \frac{\sqrt{2}}{\sqrt{1 + \sqrt{1 + s_b^2/n^2}}} \]

and

\[ u_n \equiv 1 - \frac{s_a^2 - s_b^2}{4n^2}. \]

In the limit that \( s_a \to 0 \) (i.e., the limit of a disk), Eq. (54) reduces to Eq. (19); in the limit of large \( s_b \) and \( s_a, g(s_a, s_b) \) saturates to unity.

Rearranging Eq. (52) into the form \( V = I R \) yields the plasma resistance for the ring-geometry

\[ R^\text{ring} = g(s_a, s_b) \frac{L}{\pi (b^2 - a^2) \sigma_{\parallel}}. \]

Since the specifics of the geometry in the plasma are entirely contained within the plasma resistance, all the results for the disk-shaped cathode in Secs. III and IV apply to the ring-shaped cathode with the replacement \( R_p \to R^\text{ring}. \)

VI. SELF-CONSISTENT CALCULATION OF TEMPERATURE INCREASE

The model considered in Secs. II–IV assumes constant temperature in the plasma and, thus, constant parallel conductivity. This assumption yields a linear dependence of total current on the voltage applied across the plasma. Including the self-consistent calculation of temperature introduces a non-linearity: an increase in voltage applied across the plasma causes an increase in temperature, which, in turn, increases the conductivity. This indicates that the plasma resistance is, in fact, a function of the voltage applied across the plasma \( V_p \).

The parallel and perpendicular electron heat-fluxes due to Coulomb collisions are expressed as

\[ q_\parallel = -\kappa_\parallel \nabla \cdot T_e, \quad q_\perp = -\kappa_\perp \nabla \cdot T_e, \]

where the parallel and perpendicular thermal conductivities are given by
\[ \kappa_\parallel = 3.16 \frac{n T_e e}{m_e}, \quad \kappa_\perp = 1.47 \frac{k_\parallel}{(e \Omega_e)^2}. \]  

In the bulk of the plasma, the steady-state condition for electron heat balance takes the form

\[ \nabla \cdot (\vec{q}_e + \vec{q}_\perp) = \dot{Q}_e, \]  

where the source-term \( \dot{Q}_e \) contains the Joule-heating and energy-exchange due to electron-ion Coulomb collisions

\[ \dot{Q}_e = \frac{\dot{J}_i^2}{\sigma_\perp} - \frac{3 m n}{M} \frac{\tau_e}{T_e - T_i} (T_e - T_i). \]  

Neglecting spatial gradients in the ion-temperature, the steady-state condition for ion heat-balance is simply \( \dot{Q}_i = 0 \), where \( \dot{Q}_i \) contains the Joule-heating and energy-exchange due to ion-neutral collisions and the energy-exchange due to ion-electron Coulomb collisions

\[ \dot{Q}_i = \frac{\dot{J}_i^2}{\sigma_\perp} - \frac{3 m n}{M} \frac{\tau_e}{T_e - T_i} (T_i - T_e). \]  

Summing the electron and ion equations of heat-balance and using Eq. (1) to rewrite the Joule-heating terms gives the equation of total heat balance

\[ \nabla \cdot (\vec{q}_e + \vec{q}_\perp + \vec{q}_i) = 0, \]  

where the energy-exchange due to ion-neutral collisions is assumed to be negligible.

Integrating Eq. (63) over a three-dimensional cylindrical volume spanning from cathode to anode with cross-sectional area large enough such that \( q_{\perp} \) and \( j_{\perp} \) are negligible on the radial boundary results in the expression

\[ \dot{P}_{\text{anode}} - (\dot{P}_{\text{prim}} - \dot{P}_{\text{cath}}) = I_{\text{tot}} V_p, \]  

where \( \dot{P}_{\text{prim}} - \dot{P}_{\text{cath}} \) is the power due to particles traversing the cathode sheath

\[ \dot{P}_{\text{prim}} - \dot{P}_{\text{cath}} = 2 \pi \int_0^R r \, dr \, q_{\parallel} |_{z=0}, \]  

and \( \dot{P}_{\text{anode}} \) is the power-loss to the anode due to the axial transport of energy by heat-conduction

\[ \dot{P}_{\text{anode}} = 2 \pi \int_0^R r \, dr \, q_{\parallel} |_{z=-L}. \]  

### A. Boundary conditions

The thermionic electrons injected from the cathode-surface carry their energy to the plasma with a flux given by

\[ q_{\text{eth}} |_{z=0} = j_{\text{eth}} \phi_{\text{cath}}. \]  

Ions from the plasma enter the cathode sheath at the sound speed \( c_s \), carrying their energy to the cathode-surface

\[ q_i |_{z=0} = -j_{i} \left( \frac{M c_i^2}{2e} + \phi_{\text{cath}} \right). \]  

Energetic electrons from the plasma that reach the cathode-surface carry their energy out of the plasma; integrating the appropriate moment of the Maxwellian distribution at the sheath–plasma interface gives the flux

\[ q_{\parallel} |_{z=0} = -\int d\vec{r}_\perp \int_{-\infty}^{\infty} \frac{m v^2}{2} \, \psi_f M(v) \]  

\[ = -j \left( \frac{2T_e}{e} + \phi_{\text{cath}} \right) e^{e \phi_{\text{cath}} / T_e}, \]  

where \( \psi_f M(v) = n(v) \frac{m v^2}{2} \exp \left( -\frac{m v^2}{2T_e} \right) \) is the Maxwellian velocity distribution at the sheath–plasma interface and \( v = \sqrt{v_x^2 + v_y^2} \).

Integrating Eq. (67) over the cross-sectional area of the cathode gives the total power-input due to primary heating

\[ \dot{P}_{\text{prim}} = I_{\text{eth}} \phi_{\text{cath}}. \]  

Integrating the sum of Eqs. (68) and (69) over the cross-sectional area of the cathode yields the total power lost due to particles that reach the cathode-surface

\[ \dot{P}_{\text{cath}} = I_{\text{eth}} \left( \frac{2T_e}{e} + \phi_{\text{cath}} \right) \left( 1 + e^{e \phi_{\text{cath}} / T_e} \right) - I_{\text{tot}} \frac{3T_e}{2e}. \]  

The power-loss to the anode due to the axial transport of energy may be approximated by evaluating \( q_{\parallel} \) at the end of the heated region and integrating over a cross sectional area large enough such that \( q_{\perp} \) is negligible at the radial boundary

\[ \dot{P}_{\text{anode}} = \pi R^2 \kappa_\parallel T_e. \]  

where \( \bar{L} \) is the effective length of the heated filament and \( R \) is the radius of cross-sectional area over which the integral was performed.

Equations (5), (7), (20), and (64) constitute a complete system of equations with four unknowns, \( \phi_{\text{cath}}, I_{\text{tot}}, V_p, \) and \( T_e \). Given an applied bias \( V_{\text{bias}} \) and thermionic current \( I_{\text{eth}} \), the four unknowns are obtained by solving the set of equations

\[ \phi_{\text{cath}} = \frac{T_e}{e} \Gamma + V_p - V_p, \]  

\[ I_{\text{tot}} = I_{\text{eth}} \left( 1 + e^{e \phi_{\text{cath}} / T_e} \right) + I_{\text{eth}}, \]  

\[ V_p = I_{\text{tot}} R_p, \]  

\[ \dot{P}_{\text{prim}} = \dot{P}_{\text{cath}} = I_{\text{eth}} V_p + \dot{P}_{\text{prim}}, \]

where \( \kappa_\parallel, I_{\text{tot}}, \) and \( R_p \) all depend on the temperature \( T_e \). In the case of virtual cathode formation, \( I_{\text{eth}} \) is replaced with its critical value, as given by Eq. (35).

The four equations reduce to two by eliminating \( V_p \) and \( I_{\text{tot}} \), yielding

\[ \Gamma + \psi_b - \psi_e = I_{\text{eth}} \left( 1 + e^{e \psi_e / T_e} \right) + \min \left\{ I_{\text{eth}} \frac{\psi_e}{B}, 1 \right\}, \]  

\[ 2I_{\text{eth}} \int \left[ \frac{1}{4} e^{e \psi_e / T_e} + \psi_e \right] \frac{\pi R^2}{L} e^{e^2 \psi_e / T_e} \psi_e \right] \]  

\[ = (\psi_b + \Gamma)(\psi_e - \psi_e + \Gamma), \]  

where the scaled variables are the same as those defined in Sec. III.
Equations (77) and (78) can be numerically solved using a two-dimensional Newton's method; however, a further reduction is possible. A single equation for the temperature is obtained by first solving Eq. (78) for \( \psi_c \), giving

\[
\psi_c = \beta + W\left(-\frac{2L}{\psi_b + \Gamma + 2L} e^{-\beta}\right),
\]

where

\[
\beta = \left(\frac{\psi_b + \Gamma}{L}\right) - \left(\frac{\pi R^2}{e} \frac{e^2 R p_{x||}}{2} - \frac{J_i}{J_{th}}\right).
\]

Then, inserting Eq. (79) into (77) yields a single equation for the temperature \( T_e \), which is solved numerically using a one-dimensional Newton's method.

Figure 11 shows the total current (in Amps), sheath potential drop (in Volts), and plasma temperature (in eV) self-consistently calculated for the ring cathode experiment. The cathode temperature is taken to be 1600°C, which places the threshold for virtual cathode formation, the vertical dashed line, at around 33 V. The points labeled with a black “x” denote LaB\(_6\) current measurements corresponding to the red points in Fig. 5 of the companion paper. These data points occur in the virtual cathode regime, indicating that, as long as \( J_{th} < J_{th}^{crit}(\psi_c) \), the surface temperature of the cathode is irrelevant.

For a sheath potential drop exceeding the ionization energy of helium (i.e., \( \psi_c > 24.6 \) V), the injected thermionic electrons gain enough energy to ionize neutral He atoms in the plasma. The point on the blue line in Fig. 11, corresponding to an applied bias of \( V_b \approx 48 \) V, indicates the threshold above which ionization by primaries begins to play a role. Modeling applied voltages beyond this threshold requires the self-consistent calculation of the increase in plasma density, which is not incorporated in this model.

An input parameter to this model is a spatially uniform perpendicular conductivity, which is set by a global value of the ion-neutral collision frequency, proportional to the product of the neutral density and the square root of the ion temperature. While uncertainties in measurements of ion temperature and neutral density bar precise determination of this parameter, the single value of the ion-neutral collision frequency that optimizes the agreement of this model with the ring experiment data is firmly within the expected range of values. A more detailed discussion of the sensitivity to the perpendicular conductivity is given in the experimental companion paper.

VII. CONCLUSIONS

A model for the steady-state operation of an emissive cathode in a large magnetized plasma has been presented. The model matches a global solution of the resistive plasma potential to the emissive sheath boundaries at the cathode and anode. The technique enables explicit formulas to be obtained for the total current, the partition of potential across the plasma-sheath system, and the three-dimensional current system generated in the plasma.

The model indicates that there are three facts that lead to saturation of the total current: (1) for a given applied bias, the total current cannot substantially exceed the applied bias divided by the plasma resistance; (2) the total current cannot exceed the sum of the thermionic current and ion-saturation current; (3) if the space-charge of the ion-saturation current is insufficient to cancel with the space-charge of the thermionic electrons, the injected current will be self-limited via the formation of a virtual cathode.

An analytic approximation for the threshold conditions leading to virtual cathode formation has been derived and the implications relating to experiments have been elucidated. Virtual cathode formation is linked to insufficient ion current to cancel the space-charge of the thermionic electrons, and, thus, plays an increasingly more important role as the plasma density decreases.

An extension to the model that incorporates the increase in the mean-field temperature has been derived. The self-consistent increase in temperature due to Ohmic heating increases the parallel conductivity, which, in turn, feeds back on the total current. The result is that the total current depends non-linearly on the applied voltage across the plasma, a feature corroborated by experiment.

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APPENDIX A: DERIVATION OF EQ. (17)

Evaluating the parallel derivative of Eq. (14) at \( z = 0 \) gives

\[
\frac{\partial}{\partial z} \phi(r, 0) = -\frac{2V_p}{\pi} \left[ \frac{1}{3L} \frac{1}{\sqrt{1 - (r/a)^2}} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{b_n}{\sqrt{\lambda_n^2 + b_n^2}} \right],
\]

where

\[
\lambda_n = b_n^2 + r^2/a^2 - 1,
\]

and

\[
b_n = \frac{2n}{s} \quad \text{satisfies} \quad \frac{a}{L} \sqrt{\frac{\sigma_r}{\sigma_z}}.
\]

Equation (A1) can be rewritten in the form

\[
\frac{\partial}{\partial z} \phi(r, 0) = \frac{2V_p}{\pi} \frac{a^2}{sL} \left[ \frac{1}{\sqrt{1 + \frac{4}{b_n^2}}} \right].
\]

Radially integrating Eq. (A4) from \( r = 0 \) to \( r = a \) gives

\[
\int_0^a \frac{rd\phi(r, 0)}{r} = -\frac{2V_p}{\pi} \frac{a^2}{sL} \left[ \frac{1}{\sqrt{1 + \frac{4}{b_n^2}}} \right].
\]

Rewriting the summation and multiplying by \( 2\pi\sigma_{\parallel} \) yields Eq. (17)

\[
I_{\text{tot}} = 2\pi\sigma_{\parallel} \frac{2V_p}{\pi} \frac{a^2}{sL} \left[ \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{1 + \frac{4}{b_n^2}}} \right].
\]

APPENDIX B: DERIVATION OF EQ. (24)

The transcendental equation for the total current is

\[
I_{\text{tot}} = I_1(1 - e^{\sigma_{\parallel}(\rho_0 - \rho_{\text{ina}})}) + I_{\text{eth}}.
\]

Multiplying both sides by \( e^{\sigma_{\parallel}(\rho_0 - \rho_{\text{ina}})} \), rewriting in terms of scaled variables, and rearranging give

\[
J_1 + J_{\text{eth}} - I_{\text{tot}} = J_1 e^{\sigma_{\parallel}(\rho_0 - \rho_{\text{ina}})}.
\]

Multiplying both sides by \( e^{\sigma_{\parallel}(\rho_0 - \rho_{\text{ina}})} \) arrives at the form

\[
(J_1 + J_{\text{eth}} - I_{\text{tot}}) e^{\sigma_{\parallel}(\rho_0 - \rho_{\text{ina}})} = J_1 e^{\sigma_{\parallel}(\rho_0 - \rho_{\text{ina}})}.
\]

The Lambert–W function has the property that \( W(xe^x) = x \); inserting both sides of Eq. (B3) into \( W \) gives

\[
J_1 + J_{\text{eth}} - J_{\text{tot}} = W(J_1 e^{\sigma_{\parallel}(\rho_0 - \rho_{\text{ina}})}).
\]

It is further rearranged to yield Eq. (24)

\[
J_{\text{tot}} = J_1 + J_{\text{eth}} - W(J_1 e^{\sigma_{\parallel}(\rho_0 - \rho_{\text{ina}})}).
\]

APPENDIX C: VIRTUAL CATHODE CONSTRAINT

Poisson’s equation in the cathode sheath takes the form (for the one-dimensional approximation)

\[
\frac{\partial^2 \psi}{\partial z^2} = e^{\psi} - \frac{M}{\sqrt{M^2 - 2\psi}} + \frac{1}{\sqrt{2(\psi_c + \psi)}}.
\]

where \( J = (J_{\text{eth}}/J_1) \sqrt{n/M} \) is the scaled thermionic current, \( M = u_i/e_s \), which is the Mach number of the ions at the sheath–plasma interface, and \( \lambda_{\parallel} \) is the Debye length.

Multiplying Eq. (C1) by \( \frac{\partial}{\partial z} \) and integrating with respect to \( \psi \) gives

\[
\frac{\partial^2 \psi}{\partial z^2} \left( \frac{\partial \psi}{\partial z} \right)^2 = \frac{\partial}{\partial z} \left[ e^{\psi} + M \sqrt{M^2 - 2\psi} + J \sqrt{2(\psi_c + \psi)} \right].
\]

Integrating Eq. (C2) from the sheath–plasma interface to the cathode surface yields

\[
\frac{\partial^2 \psi}{\partial z^2} \left( \frac{\partial \psi}{\partial z} \right)^2 = \frac{\partial^2 \psi}{\partial z^2} \left( \frac{\partial \psi}{\partial z} \right)^2 + B(\psi_c, M) - J \sqrt{2\psi_c},
\]

where the function

\[
B(\psi, M) = (e^{-\psi} - 1) + M \left( \frac{1}{\sqrt{1 + \frac{2\psi}{M^2}}} - 1 \right)
\]

characterizes the (non-emissive) Bohm-sheath.

The assumption of monotonicity within the sheath requires the r.h.s. of Eq. (C3) to be positive; thus, the constraint

\[
J < \frac{1}{\sqrt{2\psi_c}} \left( \frac{\partial \psi}{\partial z} \right)^2 + B(\psi_c, M)
\]

must be enforced.

The appropriate boundary condition at the sheath–plasma interface

\[
\lambda_{\parallel} \left( \frac{\partial \psi}{\partial z} \right)_{\text{sp}} = -\frac{\lambda_{\parallel}}{\lambda_{\perp}} \frac{I_{\text{tot}}}{n_a e m v_e}
\]

is taken to be small since the Debye length is much smaller than the mean-free-path of the electrons (\( \lambda_{\parallel} = 1.96 \cdot \bar{v} e \tau_e \)). Neglecting the boundary term results in the monotonicity constraint

\[
J < \frac{B(\psi_c, M)}{\sqrt{2\psi_c}}.
\]

If \( M \) is taken to be unity, Eq. (C7) reduces to
\[ J < \frac{e^{-q_e} + \sqrt{1 + 2q_e} - 2}{\sqrt{2q_e}}. \]  \hspace{1cm} (C8)

Thus, for a given potential drop across the cathode sheath, the thermionic current is constrained by Eq. (C8).

REFERENCES